Pole Assignment in Linear Systems

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Introduction

Consider a linear, time-invariant system (E, A, B):

$$E\dot{x}(t) \; = \; Ax(t) + Bu(t), \quad t \geq 0, \; x(0),$$

where
- $x(t) \in \mathbb{R}^n$ is the state
- $u(t) \in \mathbb{R}^m$ is the control
- $E, A \in \mathbb{R}^{q \times n}, \; B \in \mathbb{R}^{q \times m}, \; \operatorname{rank} B = m$

- arise in network modelling, Petri nets, composite systems...

Basic definitions: poles of the system

Definition

The pole structure of the system (E, A, B) are given by the **zero structure of the pencil $sE - A$.**

the finite zero structure

the invariant polynomials of $sE - A$, $\psi_i(s) \triangleright \psi_{i+1}(s)$

$$\begin{bmatrix} s & -1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & -1 & \\ -a_{i0} & -a_{i1} & \cdots & s - a_{il_i} \end{bmatrix}$$

the infinite zero structure

the infinite elementary divisors of $sE - A$ of orders $\mu_i := d_i + 1$, $d_i > 0$

$$\begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix}$$

Basic definitions: poles of the system

Example

$$E\dot{x}(t) \;=\; Ax(t) + Bu(t),\; t \geq 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} x(t) + Bu(t)$$

$$[sE - A] = \begin{bmatrix} s & -1 \\ 0 & s+5 \end{bmatrix} \simeq \begin{bmatrix} s(s+5) & 0 \\ 0 & 1 \end{bmatrix}$$

finite zeros at $s = 0$ and $s = -5$

invariant polynomails $\psi_1(s) = s(s+5),\; \psi_2(s) = 1.$

Conformal mapping

To deal with finite and infinite zeros in a unified way:

conformal mapping

$$s = \frac{(1 + aw)}{w}$$

- $a \in \mathbb{R}$ and is not a pole of the system.
- the point $s = \infty \Rightarrow$ the point $w = 0$
- the point $s = a \Rightarrow$ the point $w = \infty$
- the infinite zeros $\Rightarrow$ the finite zeros at $w = 0$

the pole structure of the system:

$$w^{d_1}\tilde{\psi}_1(w) \triangleright w^{d_2}\tilde{\psi}_2(w) \triangleright \ldots \triangleright w^{d_r}\tilde{\psi}_r(w)$$

the poles of the system : $\quad w^d\tilde{\psi}(w) := \prod_{i=1}^{r} w^{d_i}\tilde{\psi}_i(w)$

Pole assignment problem

Applying **the state feedback**

$$u(t) = Fx(t) + v(t),$$

where

- $F \in \mathbb{R}^{m \times n}$, and $v(t)$ is a new external input

gives the closed-loop system $(\boldsymbol{E}, \boldsymbol{A} + \boldsymbol{BF}, \boldsymbol{B})$:

$$E\dot{x}(t) = (A + BF)x(t) + Bv(t), \quad t \geq 0$$

choosing different $\boldsymbol{F}$

- alter the zero structure of $sE - A - BF$
- modify the dynamical behavior of the system

Problem Formulation

- constitutes one of the fundamental problems of control as it aims at shaping the desired system response by assigning a closed-loop poles.
- the pole assignment techniques belong to the basic tools for the controller and observers design.

Problem Formulation

Given

- a system (E, A, B),
- $\psi(s), d > 0$

Under what conditions there exists a state feedback: $\{\psi(s), d\}$ will define the poles of the closed-loop system

Example

Let the system (E, A, B) be given by

$$[sE-A, \quad -B] := \left[\begin{array}{cccc|cc} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & s & -1 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & -1 \end{array}\right],$$

and control $u(t) = Fx(t) + v(t)$,

$$F := \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix}.$$

Let $d = 2$ will be assigned.

non-unit invariant polynomials of $\tilde{D}_{EF}(w)$ and w-analogue of $sE - A - BF$ coincide for any F

$$\left[\frac{\tilde{N}_E(w)}{\tilde{D}_{EF}(w)}\right] = \left[\begin{array}{c}\tilde{N}_E(w)\\ \hline\hline \tilde{D}_{E_1}(w){\color{red}+F\tilde{N}_E(w)} \\ -------\\ \tilde{D}_{E_2}(w)\end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & (1+aw)w & 0 \\ 0 & 0 & w \\ \hline\hline 0 & (1+aw)^2 & 0 \\ 0 & 0 & 1+aw \\ - - - & - - - & - - - \\ 0 & -w^2 & 0 \\ 0 & 0 & -w \end{array}\right]$$

$$F\tilde{N}_E(w) = \begin{bmatrix} \alpha_{11} & \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ \alpha_{21} & \alpha_{22}w^2 + \alpha_{23}w & \alpha_{24}w \end{bmatrix},$$

where coefficients α_{ij} should be chosen.

Example

$$F\tilde{N}_E(w) = \begin{bmatrix} \alpha_{11} & \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ 0 & 0 & \alpha_{24}w \end{bmatrix},$$

where $\alpha_{11} \neq 0$ and the values of all other coefficients α_{ij} are irrelevant.

$$\begin{aligned} \tilde{D}_{EF}(w) &= \begin{bmatrix} \alpha_{11} & (1+aw)^2 + \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ 0 & 0 & 1 + aw + \alpha_{24}w \\ 0 & -w^2 & 0 \\ 0 & 0 & -w \end{bmatrix} \\ &\simeq \begin{bmatrix} w^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Example

$$F\tilde{N}_E(w) = \left[\begin{array}{ccc} \alpha_{11} & \alpha_{12}w^2 + \alpha_{13}w & \alpha_{14}w \\ 0 & 0 & \alpha_{24}w \end{array}\right], \ \alpha_{11} \neq 0,$$

which gives the state feedback gain

$$F = \begin{bmatrix} \alpha_{11} & \alpha_{12} - a\alpha_{13} & \alpha_{13} & \alpha_{14} \\ 0 & 0 & 0 & \alpha_{24} \end{bmatrix}$$

with $\alpha_{11} \neq 0$ (the values of other α_{ij} are irrelevant).

Example

It can be easily verified that for such an F the pencil $sE - A - BF$ has the pole at infinity of order 2,

$$[sE-A-BF] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ -\alpha_{11} & -\alpha_{12}+a\alpha_{13} & s-\alpha_{13} & -\alpha_{14} \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & s-\alpha_{24} \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & s & 0 & 0 \\ 0 & -1 & s & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & s \end{bmatrix}.$$