Summary of Probabilistic Models with Uniformly Distributed Uncertainty

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9th January 2012

Motivation

- Identification and control of the real systems
- Bayesian decision making theory
- Real system description
  - ARX model + normal noise
  - State-space model + normal noise
- Prediction and control ⇒ **estimation** and **filtering**

Motivation

ARX model + normal noise → LS
State model + normal noise → KF

⊕ Reasonable approximation of reality
⊕ Well algorithmically processed
⊖ Unsatisfactory in some applications
⊖ Problems with strictly bounded parameters

Unknown but bounded errors

⊕ Restricted support
⊖ Without statistical tools

Problem solution: **Models with uniform noise**

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Problem solution: Models with uniform noise

Underlined theory

Bayesian probabilistic approach

- model - probability density (pd)
- Bayes rule:

$$f(\mathbf{X}|data) \propto f(data|\mathbf{X})f(\mathbf{X})$$

MAP estimation

$$\hat{\mathbf{X}} = \arg\max_{\mathbf{X}^*} f(data|\mathbf{X})f(\mathbf{X})$$

Uniform ARX model - description

$$y_t = \psi_t'\theta + e_t$$

t - discrete time, $t \in t^* = 1, 2, ..., T$
y_t - measured output
$\psi_t = [y_{t-1}, \dots, y_{t-n}, u_t, \dots, u_{t-n}]$ - regression vector,
$\theta = [a_1, \dots, a_n, b_0, \dots, b_n]$ - regression coefficients,
$e_t \sim \mathcal{U}(-r, r)$ - measurement noise.

ARX model - parameter estimation

Parameters $\Theta = (\theta, r)$

$$f(\Theta|\text{data}) \propto \frac{1}{r^{\nu_t}} \chi(\mathcal{M})$$

$$\mathcal{M} : \begin{cases} \text{prior information} \\ \text{ARX model \& data} \end{cases}$$

Statistics:

counter $\nu_t = \nu_{t-1} + 1$

data matrix $W'_t = \left[W'_{t-1}, \Psi_t\right]$

Ψ_t - data vector; $\Psi'_t \equiv [y_t, \psi'_t]$

ARX model - approximation

Point MAP estimate $\rightarrow$ linear programming (LP)

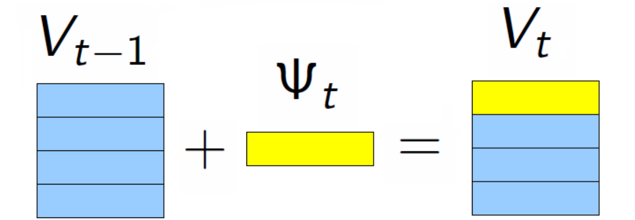
- size of data matrix W_t increases with the time
  $\Rightarrow$ recursive estimation needs **approximation**
- original $W_t \rightarrow$ approximated V_t

Problems solved:

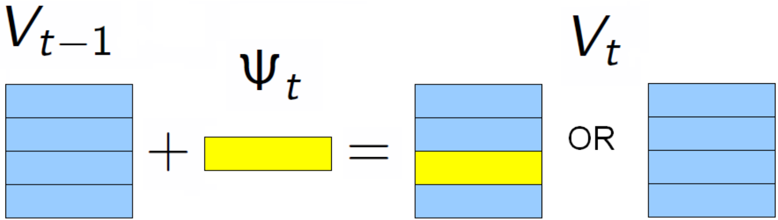
- choice of size of matrix $V_t \Leftrightarrow$ memory length
- update and approximation:
  $V_{t-1} + \Psi_t \rightarrow V_t$

ARX model - approximation - variants

First in - first out principle:

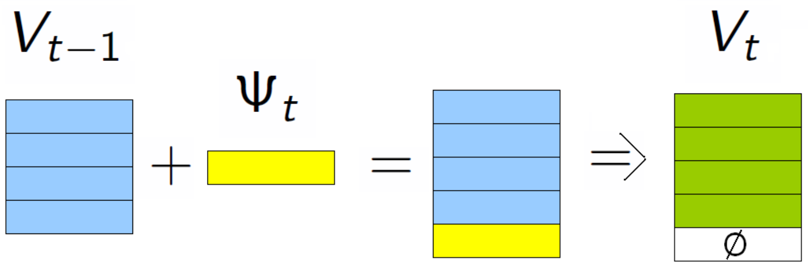


Removal of the least informative data:



ARX model - approximation - variants

Circumscribing:



State-space model with uniform noise (SU model)

For $t \in t^* = 1, 2, \ldots, T$

$$
\begin{aligned}
\mathbf{x}_t &= g(\mathbf{x}_{t-1}, \mathbf{u}_t) + \mathbf{w}_t; \qquad & f(\mathbf{w}_t|\mathbf{q}) = \mathcal{U}(-\mathbf{q}, \mathbf{q}) \\
\mathbf{y}_t &= h(\mathbf{x}_t) + \mathbf{e}_t; \qquad & f(\mathbf{e}_t|\mathbf{r}) = \mathcal{U}(-\mathbf{r}, \mathbf{r})
\end{aligned}
$$

$\mathbf{u}_t$ - input
$\mathbf{x}_t$ - state
$\mathbf{y}_k$ - output
g, h - real vector functions
$\mathbf{w}_t$, $\mathbf{e}_t$ - state and output noises

SU model - pdf representation

$$\mathbf{X} = \left[\ \mathbf{x}'_{t-\Delta} \quad \dots \quad \mathbf{x}'_t \quad \mathbf{q}' \quad \mathbf{r}'\ \right]'$$

$$f(\mathbf{X}|data) \propto \prod_{i=1}^{m} q_i^{-(\Delta+1)} \prod_{j=1}^{n} r_j^{-(\Delta+1)} \ \chi(\mathcal{S})$$

$$\mathcal{S} : \begin{cases} \text{prior information} \\ \text{state-space model \& data} \\ \text{restriction on states} \end{cases}$$

SU model - estimation

The MAP estimate of **X**:

$$\hat{\mathbf{X}} = \arg\min_{\mathbf{X}\in\mathcal{S}} \left(\sum_{i=1}^{m} \ln(q_i) + \sum_{j=1}^{n} \ln(r_j) \right)$$

The MAP estimate $\rightarrow$ non-linear programming form

SU model - variants

- SU model with missing data
- linear SU model with unknown model matrices
- linear SU model with correlated noise

SU model - estimates characteristics

Window $\Delta \Rightarrow$ multiple state estimates

time	estimates						
t:			$\hat{\mathbf{x}}_t$	$\hat{\mathbf{x}}_{t-1}$	$\dots$		$\hat{\mathbf{x}}_{t-\Delta}$
$t+1$:		$\hat{\mathbf{x}}_{t+1}$	$\hat{\mathbf{x}}_t$	$\dots$	$\hat{\mathbf{x}}_{t-\Delta+1}$		
$\vdots$			$\vdots$				
$t+\Delta$:	$\hat{\mathbf{x}}_{t+\Delta}$	$\dots$	$\hat{\mathbf{x}}_t$				

Model:

$$f\left(\hat{x}_{t|t}, \dots, \hat{x}_{t|t+\Delta} \middle| x_t, \rho\right)$$

“data”

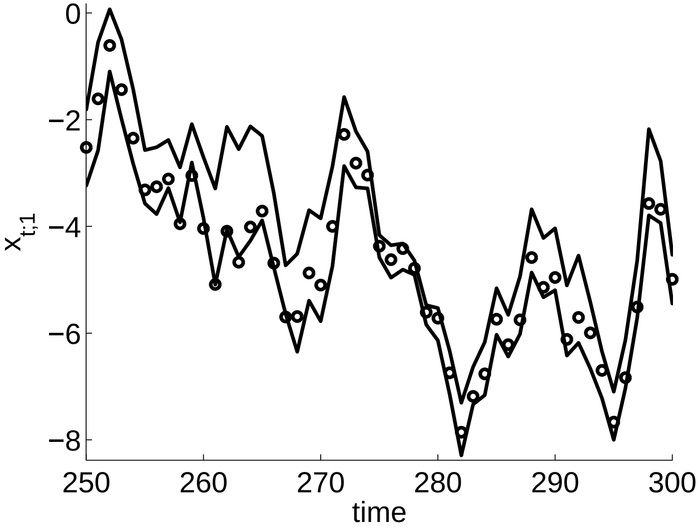
- $\hat{x}_{t|t}, \ldots, \hat{x}_{t|t+\Delta}$

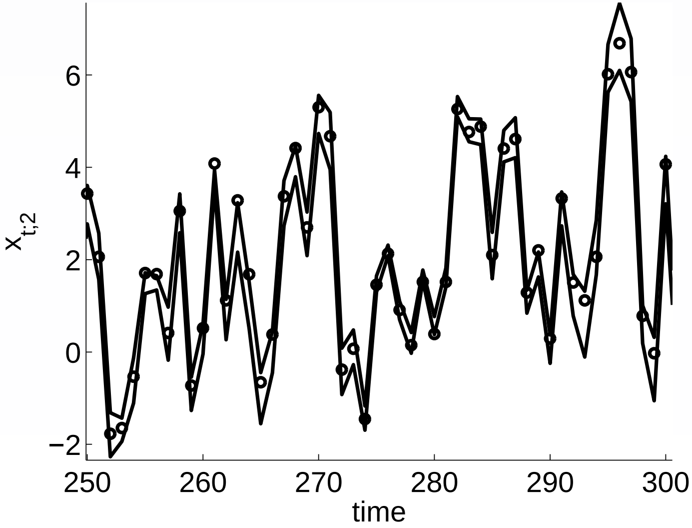
statistics

- $\underline{s} = \min \left\{ \hat{x}_{t|k} \right\}$
- $\overline{s} = \max \left\{ \hat{x}_{t|k} \right\}$
- n

interval estimate

- $[E[x_t - \rho|\underline{s}, \overline{s}, n],\ E[x_t + \rho|\underline{s}, \overline{s}, n]]$


0
−2
−4
−6
−8
$x_{t;1}$
250
260
270
280
290
300
time

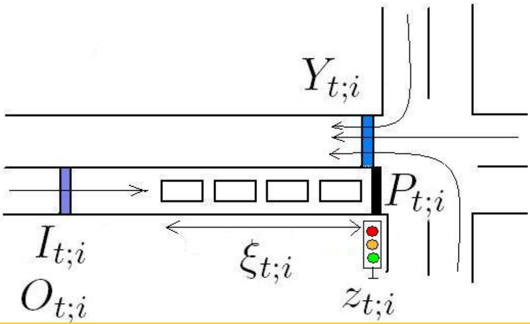


$x_{t;2}$
6
4
2
0
−2
250
260
270
280
290
300
time

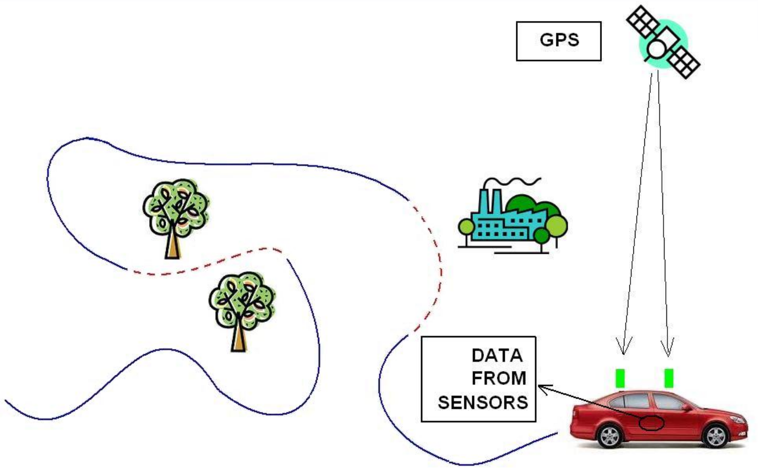
Application - Queue length estimation

Model of **controlled intersection** - quantities:

- measured - intensity I_t and Y_t, occupancy O_t
- estimated - length of the car queue ξ_t, parameters κ, β, λ
- given - green time z_t, sat. flow S, turning rates α



Application - Estimation of moving vehicle position



Conclusion - Benefits of uniform models

They

- allow estimation of the noise range
- respect hard bounds on the estimated quantities
- enable the joint estimation of parameters, states, and noise bounds
- fit to robust-control applications
- provide an easy entry of the partial knowledge on the model matrices
- update estimates on the whole window of the length Δ
- enable parameter tracking

Thank you for your attention!