Synthesis of anisotropy-based robust stochastic controllers by convex optimization and semidefinite programming

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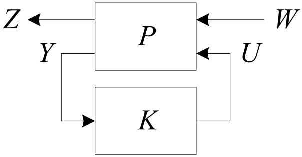
- Introduction
- Background
- Bounded Real Lemma for anisotropic norm
- Synthesis of anisotropic suboptimal controllers by convex optimization
- Multiobjective and robust problems of anisotropic suboptimal controller design
- Numerical examples
- Conclusion

- Standard linear plant model P
- Linear controller K
- External disturbance W
- Control U
- Controlled output Z
- Measurement Y



Controller synthesis problem

Find controller K to stabilize closed-loop system and minimize some performance criterion

$$J(K) \to \min_{K \in \mathcal{K}}$$

$\diamond$ LQG ($\mathcal{H}_2$) problem, W is Gaussian white noise: $\mathbf{E}|z_k|^2 \to \min_{K \in \mathcal{K}}$

$\diamond$ $\mathcal{H}_\infty$ optimization problem, $W \in \ell_2$: $\|T_{zw}\|_\infty \to \min_{K \in \mathcal{K}}$

or (suboptimal problem) $\|T_{zw}\|_\infty < \gamma$

$\mathcal{H}_2$ norm:

$$\|F\|_2 := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr}(\widehat{F}(\omega)\widehat{F}^*(\omega))d\omega \right)^{1/2}$$

$\mathcal{H}_\infty$ norm:

$$\|F\|_\infty := \sup_{|z|<1} \overline{\sigma}(F(z)) = \operatorname{ess\,sup}_{\omega\in\Omega} \overline{\sigma}(\widehat{F}(\omega))$$

$$\widehat{F}(\omega) := F(\mathrm{e}^{i\omega}) \qquad \omega \in \Omega := [-\pi, \pi] \qquad \overline{\sigma}(F) := \sqrt{\lambda_{\max}(F^*F)}$$

Similarity

- Riccati equations
- $\gamma \to +\infty$, $\mathcal{H}_\infty$ Riccati equations $\to \mathcal{H}_2$ (LQG)

Difference

- $\mathcal{H}_\infty$: maximum of gain-frequency response $\to$ min
- $\mathcal{H}_2$: average amplitude over all frequences $\to$ min

Disturbance assumptions

- $\mathcal{H}_2$ (LQG) : Gaussian white noise
- $\mathcal{H}_\infty$: square integrable (summable) signal

If disturbance signal assumption violates

- $\mathcal{H}_2$ (LQG) : closed-loop system can not perform well if disturbance is "far from" Gaussian white noise
- $\mathcal{H}_\infty$: control can be very conservative if input signal is "close enough" to Gaussian white noise

Trade-offs between $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems

- suboptimal $\mathcal{H}_\infty$ controller is not unique
  $\Rightarrow$ additional performance criterion can be considered
- natural choice: $\mathcal{H}_2$ norm of closed-loop transfer function

[Bernstein, Haddad, 1989]

$$\|T_{zw}\|_\infty < \gamma$$

$$\|T_{zw}\|_2 \to \min$$

[Mustafa, Glover, 1991]

$$\|T_{zw}\|_\infty < \gamma$$

$$\sup \|T_{zw}\|_2 \to \min$$

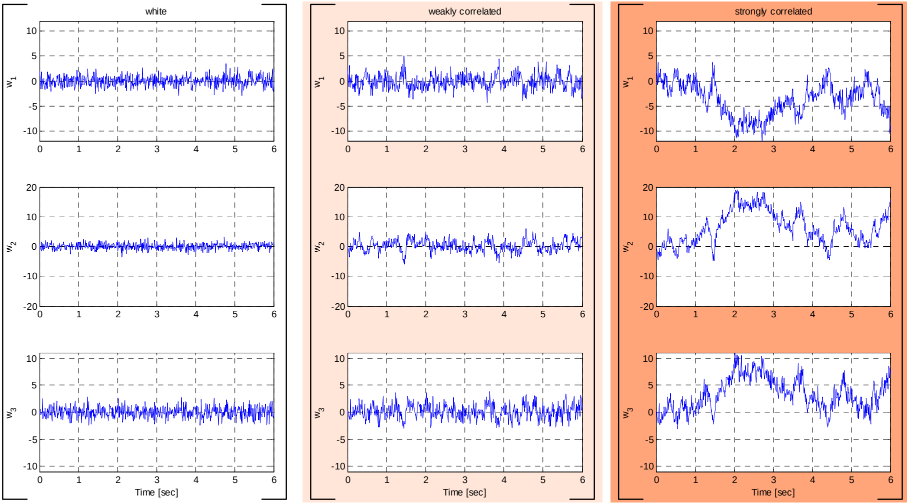
Suboptimal $\mathcal{H}_\infty$ controller minimizing $\mathcal{H}_\infty$ entropy functional [Mustafa & Glover, 1991]

$$\|F\|_\infty < \gamma$$

$$J(\gamma, F) := -\frac{\gamma^2}{2\pi} \int_{-\pi}^{\pi} \ln\det\left(I_m - \gamma^{-2}\widehat{F}^*(\omega)\widehat{F}(\omega)\right) d\omega \to \min$$

Some features

- $J(\gamma, F) \geqslant \|F\|_2^2$
- $\mathcal{H}_\infty$ entropy minimization is equivalent to risk sensitivity problem
- resulting (central) controller is unique


white
weakly correlated
strongly correlated
w_1
w_2
w_3
Time [sec]

Relative entropy (Kullback-Leibler informational divergence) [Cover & Thomas, 1991], [Gray, 1991]

$$\mathbf{D}(P\|M) := \begin{cases} \mathbf{E}\ln\dfrac{dP}{dM} & \text{if } P \ll M \\ +\infty & \text{otherwise} \end{cases}$$

- P, M : probability measures on measurable space $(\Omega, \mathfrak{F})$
- $\mathbf{E}$: taken over P
- $dP/dM : \Omega \to \mathbb{R}_+$ is the Radon-Nikodym derivative

Recall [Gray, 1991]

$$\Omega = \mathbb{R}^m \qquad P, M \ll \text{mes}_m \text{ with PDFs } f,\ g$$

$$\mathbf{D}(P\|M) = \int_{\mathbb{R}^m} f(x)\ln\frac{f(x)}{g(x)}dx$$

Differential entropy [Cover & Thomas, 1991]

$$\mathbf{h}(X) := -\mathbf{E}\ln f(X) = -\int_{\mathbb{R}^m} f(x)\ln f(x)dx$$

- X : random vector
- f : PDF

Mutual information [Gray, 1991]

$$\mathbf{I}(X;Y) := \int_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}} f_{12}(x,y)\ln\frac{f_{12}(x,y)}{f_1(x)f_2(y)}dxdy$$

- X, Y : random vectors
- f_1, f_2 : marginal PDFs
- f_{12} : joint PDF

Gaussian PDF with zero mean and covariance matrix λI_m:

$$p_{m,\lambda}(w) := (2\pi\lambda)^{-m/2} \exp\left(-\frac{|w|^2}{2\lambda}\right) \qquad w \in \mathbb{R}^m$$

For any $W \in \mathbb{L}_2^m$ with PDF $f : \mathbb{R}^m \to \mathbb{R}_+$

$$\mathbf{D}(f \| p_{m,\lambda}) = \mathbf{E} \ln \frac{f(W)}{p_{m,\lambda}(W)} = \frac{m}{2} \ln(2\pi\lambda) + \frac{\mathbf{E}|W|^2}{2\lambda} - \mathbf{h}(W),$$

$$\mathbf{h}(X) := -\mathbf{E} \ln f(X) = -\int_{\mathbb{R}^m} f(x) \ln f(x) dx$$

Anisotropy of random vector [Vladimirov et al., 2006]

$$\mathbf{A}(W) := \min_{\lambda > 0} \mathbf{D}(f \| p_{m,\lambda}) = \frac{m}{2} \ln\left(\frac{2\pi e}{m} \mathbf{E}|W|^2\right) - \mathbf{h}(W)$$

- minimum is achieved at $\lambda = \mathbf{E}|W|^2/m$

$\mathbb{G}^m(\mu, \Sigma)$: class of Gaussian vectors $W \in \mathbb{R}^m$ with PDF

$$p(w) := (2\pi)^{-m/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}\|w-\mu\|^2_{\Sigma^{-1}}\right) \qquad \|w\|_M := \sqrt{w^{\mathrm{T}}Mw}$$

Anisotropy: Basic properties [Vladimirov et al., 2006]

1. $\mathbf{A}(\sigma UW) = \mathbf{A}(W)$ for any orthogonal $U \in \mathbb{R}^{m\times m}$ and $\sigma \in \mathbb{R}\setminus\{0\}$
2. for any $(m\times m)$-matrix $\Sigma \succ 0$

$$\min\left\{\mathbf{A}(W) : W \in \mathbb{L}_2^m,\ \mathbf{E}(WW^{\mathrm{T}}) = \Sigma\right\} = -\frac{1}{2}\ln\det\frac{m\Sigma}{\operatorname{tr}\Sigma}$$

   minimum is only achieved at $W \in \mathbb{G}^m(0, \Sigma)$
3. $\mathbf{A}(W) \geqslant 0$ for any $W \in \mathbb{L}_2^m$, and $\mathbf{A}(W) = 0$ if and only if $W \in \mathbb{G}^m(0, \lambda I_m)$ for some $\lambda > 0$

Discrete-time random signal $W := (w_k)_{-\infty<k<+\infty}$: stationary sequence of random vectors $w_k \in \mathbb{L}_2^m$

On time interval $[s, t]$:

$$W_{s:t} := \begin{bmatrix} w_s \\ \vdots \\ w_t \end{bmatrix}$$

Assumption: $W_{0:N}$ is distributed absolutely continuously $\forall N \geqslant 0$

Mean anisotropy of random sequence [Vladimirov et. al, 2006]

$$\overline{\mathbf{A}}(W) := \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N})}{N}$$

Mean anisotropy and mutual information [Vladimirov et al., 2006]

$$\overline{\mathbf{A}}(W) = \mathbf{A}(w_0) + \mathbf{I}(w_0; (w_k)_{k<0})$$

where

$$\mathbf{I}(w_0; (w_k)_{k<0}) := \lim_{s \to -\infty} \mathbf{I}(w_0; W_{s:-1})$$

For Gaussian random sequence W

$$\mathbf{I}(w_0; (w_k)_{k<0}) = \frac{1}{2} \ln \det \left(\mathbf{cov}(w_0)\, \mathbf{cov}(\widetilde{w}_0)^{-1} \right)$$

where

$$\widetilde{w}_0 := w_0 - \mathbf{E}(w_0 \mid (w_k)_{k<0})$$

Let $V := (v_k)_{-\infty<k<+\infty}$, $v_k \in \mathbb{G}^m(0, I_m)$. Suppose $W = GV$:

$$w_j = \sum_{k=0}^{+\infty} g_k v_{j-k} \qquad -\infty < j < +\infty$$

Spectral density of W: $\qquad S(\omega) := \widehat{G}(\omega)\widehat{G}^*(\omega) \qquad -\pi \leqslant \omega < \pi$

$$\widehat{G}(\omega) := G(\mathrm{e}^{i\omega}) \qquad G(z) := \sum_{k=0}^{+\infty} g_k z^k \in \mathcal{H}_2^{m\times m}$$

In terms of S and G [Vladimirov et al., 1996]

$$\overline{\mathbf{A}}(W) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\frac{mS(\omega)}{\|G\|_2^2}d\omega$$

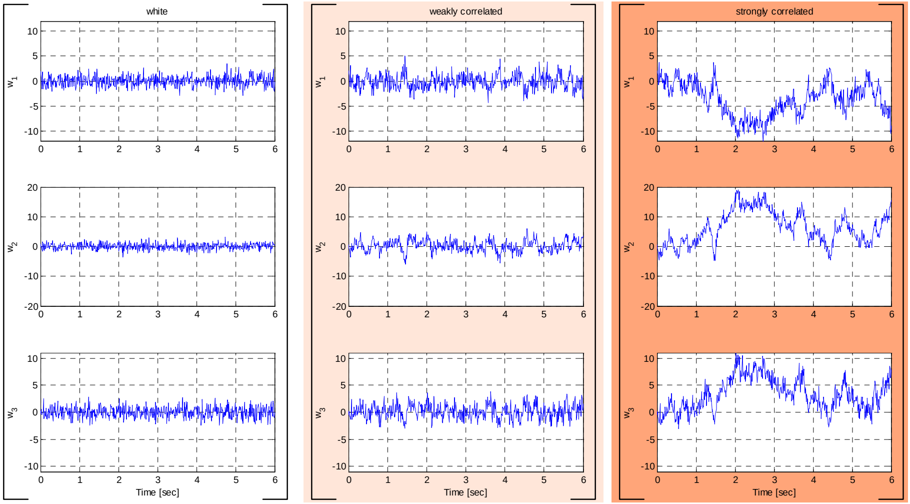
Basic properties of mean anisotropy:

1. $0 \leqslant \overline{\mathbf{A}}(W) < +\infty$ if rank $\widehat{G}(\omega) = m$ for almost all $\omega \in [-\pi, \pi)$
2. $\overline{\mathbf{A}}(W) = +\infty$ otherwise
3. $\overline{\mathbf{A}}(W) = 0$ if and only if G is an all-pass system up to a nonzero constant factor, in which case

$$S(\omega) = \lambda I_m \qquad -\pi \leqslant \omega < \pi$$

for some $\lambda > 0$, so that $W \in \mathbb{G}^m(0, \lambda I_m)$

$\overline{\mathbf{A}}(W) = 0$ $\overline{\mathbf{A}}(W) = 1$ $\overline{\mathbf{A}}(W) = 6$



$F \in \mathcal{H}_\infty^{p\times m}$: LDTI system with input $W = GV$, $V \in \mathbb{G}^m(0, \lambda I_m)$, and output $Z = FW$

Denote

$$\mathcal{G}_a := \{G \in \mathcal{H}_2^{m\times m} : \overline{\mathbf{A}}(G) \leqslant a\} \qquad \mathcal{W}_a := \{W \in \ell_{\mathcal{P}}^m : \overline{\mathbf{A}}(W) \leqslant a\}$$

$$\|W\|_{\mathcal{P}} := \left(\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \mathbf{E}|w_k|^2\right)^{1/2} = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr} S(\omega)d\omega\right)^{1/2}$$

Anisotropic norm of LDTI system [Vladimirov et al., 1996]

$$|||F|||_a := \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2} = \sup_{W\in\mathcal{W}_a} \frac{\|Z\|_{\mathcal{P}}}{\|W\|_{\mathcal{P}}}$$

- Important properties of anisotropic norm:

$$\frac{1}{\sqrt{m}}\|F\|_2 = |||F|||_0 \leqslant \lim_{a\to+\infty} |||F|||_a = \|F\|_\infty \qquad (1)$$

$$|||F|||_a < +\infty \qquad \Leftrightarrow \qquad \text{system } F \text{ is stable}$$

How to calculate the anisotropic norm in state space? [Vladimirov et al., 1996]

$$F \in \mathcal{H}_\infty^{p\times m}: \quad \begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}, \quad -\infty < k < +\infty$$

$$|||F|||_a = \left(\frac{1}{q} \left(1 - \frac{m}{\mathrm{tr}(LPL^{\mathrm{T}} + \Sigma)} \right) \right)^{1/2}$$

$$\begin{aligned} R &= A^{\mathrm{T}}RA + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L \\ L &:= \Sigma(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C) \\ \Sigma &:= (I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1} \end{aligned}$$

$$P = (A+BL)P(A+BL)^{\mathrm{T}} + B\Sigma B^{\mathrm{T}}$$

$$a = -\frac{1}{2}\ln\det\left(\frac{m\,\Sigma}{\mathrm{tr}(LPL^{\mathrm{T}} + \Sigma)} \right)$$

Most general form of linear matrix inequality:

$$F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \succcurlyeq 0$$

- $F_i = F_i^{\mathrm{T}}$ are given
- x_i are decision variables
- $\succcurlyeq 0$ stands for positive semidefinite (nonnegative eigenvalues)
- actually affine matrix constraint
- convex feasible set $\{x \in \mathbb{R}^n : F(x) \succcurlyeq 0\}$
- strict version $F(x) \succ 0$ means strictly positive eigenvalues

Traditional form of LMIs in control theory (e.g. Lyapunov inequality):

$$\exists P = P^{\mathrm{T}} \succ 0 : \qquad A^{\mathrm{T}}P + PA \prec 0$$

$$\Updownarrow$$

system $\dot{x}(t) = Ax(t)$ is stable (all trajectories converge to zero)

Most important instruments:

Schur theorem (complement)

$$\begin{bmatrix} A(x) & B(x) \\ B(x)^{\mathrm{T}} & C(x) \end{bmatrix} \succcurlyeq 0, \quad C(x) \succ 0 \quad \Leftrightarrow \quad A(x) - B(x)C(x)^{-1}B(x)^{\mathrm{T}} \succcurlyeq 0, \quad C(x) \succ 0$$

Elimination (projection) lemma

$$A(x) + B(x)XC(x)^{\mathrm{T}} + C(x)X^{\mathrm{T}}B^{\mathrm{T}}(x) \succ 0 \quad \Leftrightarrow \quad \begin{array}{c} W_B^{\mathrm{T}}(x)A(x)W_B(x) \succ 0 \\ W_C^{\mathrm{T}}(x)A(x)W_C(x) \succ 0 \end{array}$$

- W_B, W_C are orthogonal complements of B, C
- x independent of matrix X

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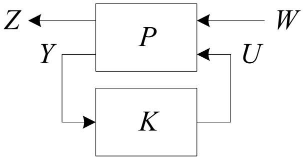


Suboptimal synthesis problem

Given $a \geqslant 0$, $\gamma > 0$, find a controller K that stabilizes $\mathcal{F}_l(P, K)$ s.t.

$$\|T_{zw}\|_a < \gamma \qquad (2)$$

Analysis problem

Given $a \geqslant 0$, $\gamma > 0$, verify if (2) holds true

Theorem 1

$$F \in \mathcal{H}_\infty^{p\times m}: \quad \begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \tag{3}$$

Given $a \geqslant 0$, $\gamma > 0$,

$$|||F|||_a < \gamma$$

if and only if there exists $\eta > \gamma^2$ such that

$$\eta - \left(\mathrm{e}^{-2a} \det\left(\eta I_m - \mathcal{B}^\mathrm{T}\Phi\mathcal{B} - \mathcal{D}^\mathrm{T}\mathcal{D}\right)\right)^{1/m} < \gamma^2 \tag{4}$$

holds true for a real $(n \times n)$-matrix $\Phi = \Phi^\mathrm{T} \succ 0$,

$$\begin{bmatrix} \mathcal{A}^\mathrm{T}\Phi\mathcal{A} - \Phi + \mathcal{C}^\mathrm{T}\mathcal{C} & \mathcal{A}^\mathrm{T}\Phi\mathcal{B} + \mathcal{C}^\mathrm{T}\mathcal{D} \\ \mathcal{B}^\mathrm{T}\Phi\mathcal{A} + \mathcal{D}^\mathrm{T}\mathcal{C} & \mathcal{B}^\mathrm{T}\Phi\mathcal{B} + \mathcal{D}^\mathrm{T}\mathcal{D} - \eta I_{m_w} \end{bmatrix} \prec 0 \tag{5}$$

Corollary: computing anisotropic norm. Denote $\widehat{\gamma} := \gamma^2$

$$\widehat{\gamma} \longrightarrow \inf \quad \text{over} \quad \eta, \Phi, \widehat{\gamma} \quad \text{that satisfy (4), (5)}$$

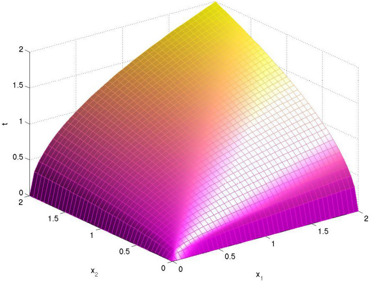
$$|||F|||_a = \sqrt{\widehat{\gamma}_\star}$$

- $-\det(\Psi)^{1/m}$, $\Psi \in \mathbb{R}^{m\times m}$, $\Psi = \Psi^{\mathrm{T}} \succ 0$ — convex function
- $\det(\Psi)^{1/m} = \sqrt[m]{\lambda_1(\Psi)\cdot\ldots\cdot\lambda_m(\Psi)}$ — geometric mean of eigenvalues
- Geometric mean of two nonnegative variables: the set

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ t \end{bmatrix} \in \mathbb{R}^3 \colon x_1, x_2 \geqslant 0, \sqrt{x_1 x_2} \geqslant 0 \right\}$$

is representable as a second-order cone (SOC) [Ben-Tal, Nemirovskii, 2000]:

$$\exists u \colon u \geqslant t, \left\| \begin{bmatrix} u \\ \frac{x_1 - x_2}{2} \end{bmatrix} \right\| \leqslant \frac{x_1 + x_2}{2}$$



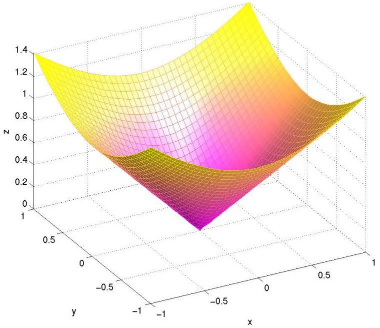
- Geometric mean of $2k$ nonnegative variables: the set

$$\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_{2k} \\ t \end{bmatrix} \in \mathbb{R}^{2k+1} \colon x_i \geqslant 0, \sqrt[2k]{x_1\cdot\ldots\cdot x_{2k}} \geqslant 0 \right\}$$

is also SOC-representable (iteratively)

- SOC is LMI-representable [Lobo, Vandenberghe, Boyd, Lebret, 1997]

- Example: the Lorentz cone



$$\mathcal{L} = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} : \; \|x\| \leqslant t \right\}$$

can be represented as

$$\mathcal{L} = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \in \mathbb{R}^{n+1} : \; \begin{bmatrix} tI_n & x \\ x^{\mathrm{T}} & t \end{bmatrix} \succcurlyeq 0 \right\}$$

- $\{\Psi = \Psi^{\mathrm{T}} \succ 0 : \; -\det(\Psi)^{1/m} \leqslant t\}$ can be represented by

$$\left\{ \Psi = \Psi^{\mathrm{T}} \succ 0 : \; \begin{bmatrix} X & \Delta \\ \Delta^{\mathrm{T}} & \operatorname{diag} \Delta \end{bmatrix} \succcurlyeq 0, \;\; t \leqslant (\delta_1 \cdot \ldots \cdot \delta_m)^{1/m} \right\}$$

where Δ is a lower triangular matrix of additional variables with diagonal elements δ_i; $t \leqslant (\delta_1 \cdot \ldots \cdot \delta_m)^{1/m}$ (hypograph of a concave monomial) is conic representable [Ben-Tal, Nemirovskii, 2000]

Numerical solution: interface YALMIP [Löfberg, 2004], function `geomean`, solver SeDuMi [Sturm, 1999] in Matlab or Scilab

SeDuMi = Self-Dual-Minimization: implements self-dual embedding technique for optimization over self-dual homogeneous cones

- Y.Ye, M.J. Todd, and S. Mizuno. An $O(\sqrt{n}L)$-iteration homogeneous and self-dual linear programming algorithm // *Mathematics of Operations Research*, 19:53–67, 1994.
- Y. Nesterov and M.J. Todd. Self-scaled barriers and interior-point methods for convex programming // *Mathematics of Operations Research*, 22(1):1–42, 1997.
- J.F. Sturm. Primal-Dual Interior Point Approach to Semidefinite Programming, volume 156 of *Tinbergen Institute Research Series*. Thesis Publishers, Amsterdam, The Netherlands, 1997.

$$\eta - \left(\mathrm{e}^{-2a}\det\left(\eta I_m - \mathcal{B}^{\mathrm{T}}\Phi\mathcal{B} - \mathcal{D}^{\mathrm{T}}\mathcal{D}\right)\right)^{1/m} < \gamma^2$$

$$\begin{bmatrix} \mathcal{A}^{\mathrm{T}}\Phi\mathcal{A} - \Phi + \mathcal{C}^{\mathrm{T}}\mathcal{C} & \mathcal{A}^{\mathrm{T}}\Phi\mathcal{B} + \mathcal{C}^{\mathrm{T}}\mathcal{D} \\ \mathcal{B}^{\mathrm{T}}\Phi\mathcal{A} + \mathcal{D}^{\mathrm{T}}\mathcal{C} & \mathcal{B}^{\mathrm{T}}\Phi\mathcal{B} + \mathcal{D}^{\mathrm{T}}\mathcal{D} - \eta I_{m_w} \end{bmatrix} \prec 0$$

$a \to 0 : \lim\limits_{a\to 0} |||F|||_a = \frac{1}{\sqrt{m}}\|F\|_2$

$$\mathcal{A}^{\mathrm{T}}\Phi\mathcal{A} - \Phi + \mathcal{C}^{\mathrm{T}}\mathcal{C} \prec 0$$

$$\mathrm{tr}(\mathcal{B}^{\mathrm{T}}\Phi\mathcal{B} + \mathcal{D}^{\mathrm{T}}\mathcal{D}) < m\gamma^2$$

$$\Updownarrow$$

$$\|F\|_2 < \sqrt{m}\gamma$$

$a \to +\infty : \lim\limits_{a\to +\infty} |||F|||_a = \|F\|_\infty$

$$\begin{bmatrix} \mathcal{A}^{\mathrm{T}}\bar{\Phi}\mathcal{A} - \bar{\Phi} & \mathcal{A}^{\mathrm{T}}\bar{\Phi}\mathcal{B} & \mathcal{C}^{\mathrm{T}} \\ \mathcal{B}^{\mathrm{T}}\bar{\Phi}\mathcal{A} & \mathcal{B}^{\mathrm{T}}\bar{\Phi}\mathcal{B} - \gamma I_m & \mathcal{D}^{\mathrm{T}} \\ \mathcal{C} & \mathcal{D} & -\gamma I_p \end{bmatrix} \prec 0$$

$$\Updownarrow$$

$$\|F\|_\infty < \gamma$$

Theorem 2 (BRL in reciprocal matrices)

Given $F \in \mathcal{H}_\infty^{p\times m}$ with realization (3), $a \geqslant 0$, $\gamma > 0$,

$$|\!|\!| F |\!|\!|_a < \gamma$$

if there exists $\eta > \gamma^2$ such that

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m} < \gamma^2 \tag{6}$$

holds true for real $(m \times m)$-matrix $\Psi = \Psi^{\mathrm{T}} \succ 0$ and $(n \times n)$-matrix $\Phi = \Phi^{\mathrm{T}} \succ 0$ that satisfy

$$\begin{bmatrix} \Psi - \eta I_m & \mathcal{B}^{\mathrm{T}} & \mathcal{D}^{\mathrm{T}} \\ \mathcal{B} & -\Phi^{-1} & 0 \\ \mathcal{D} & 0 & -I_p \end{bmatrix} \prec 0 \quad \begin{bmatrix} -\Phi & 0 & \mathcal{A}^{\mathrm{T}} & \mathcal{C}^{\mathrm{T}} \\ 0 & -\eta I_m & \mathcal{B}^{\mathrm{T}} & \mathcal{D}^{\mathrm{T}} \\ \mathcal{A} & \mathcal{B} & -\Phi^{-1} & 0 \\ \mathcal{C} & \mathcal{D} & 0 & -I_p \end{bmatrix} \prec 0 \tag{7}$$

Let $\Pi := \Phi^{-1}$

$$\|F\|_a < \gamma$$

$$\Uparrow$$

find $\eta > \gamma^2$, matrix $\Psi \succ 0$ and reciprocal matrices $\Phi \succ 0$, $\Pi \succ 0$, $\Phi\Pi = I_n$ that satisfy LMI (7) under constraint (6) or determine insolvability

Examples of algorithms for searching reciprocal matrices:

1. Apkarian P. and Tuan H.D. Concave programming in control theory // J. of Glob. Opt., 1999, Vol. 15, p. 343–370.
2. Balandin D.V. and Kogan M.M. Synthesis of controller on the base of a solution of linear matrix inequalities and a search algorithm for reciprocal matrices // Automation and Remote Contr., 2005, Vol. 66, No. 1, p. 74–91.
3. Polyak B.T. and Gryazina E.N. Hit-and-Run: Randomized technique for control problems recasted as concave programming // Proc. 18th IFAC World Congr., Milano, Italy, 2011.

Algorithm on the base of conditional gradient method for concave function [Apkarian & Tuan, 1999]:

1. $k = 0$. Choose initial values $\Pi_0 > 0$, $\Phi_0 > 0$, which satisfy (7) for some Ψ, η
2. Solve local convex program:

$$\begin{gathered} \mathrm{tr}\,(\Pi + \Phi_k^{-1}\Phi\Phi_k^{-1}) \to \min \\ \text{over} \quad \Psi, \Phi, \Pi, \eta \quad \text{that satisfy (7) and} \begin{bmatrix} \Pi & I \\ I & \Phi \end{bmatrix} \succcurlyeq 0 \end{gathered} \tag{8}$$

3. If (8) is solvable and unknown variables are found, verify stop conditions

$$|\,\mathrm{tr}\,(\Pi + \Phi_k^{-1}\Phi\Phi_k^{-1})| < \epsilon \qquad k_N \leqslant k \tag{9}$$

   ϵ — given accuracy, k_N — given maximum number of iterations. One of conditions (9) holds, algorithm stops
4. $k := k + 1$; $\Phi_k := \Phi$; go to step 2

Standard LDTI state-space model:

$$P(z): \begin{bmatrix} x_{k+1} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ I_{n_x} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix}$$

$y_k = x_k \in \mathbb{R}^{n_x}$, $z_k \in \mathbb{R}^{p_z}$, $w_k \in \mathbb{R}^{m_w}$, $u_k \in \mathbb{R}^{m_u}$, $\overline{\mathbf{A}}(W) \leqslant a$

Problem 1. Synthesis of state-feedback anisotropic controller

Given $a \geqslant 0$ $\gamma > 0$, find a state-feedback controller

$$u_k = Kx_k$$

that stabilizes closed-loop system $T_{zw}(z)$ with realization

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{c|c} A + B_uK & B_w \\ \hline C_z + D_{zu}K & D_{zw} \end{array}\right]$$

i.e. ensures $\rho(A + B_uK) < 1$, such that $|||T_{zw}|||_a < \gamma$

Theorem 3. Solution to state-feedback controller synthesis problem

Given $a \geqslant 0$ and $\gamma > 0$, anisotropic controller $u_k = Kx_k$ that ensures $\rho(A + B_u K) < 1$ and $\|T_{zw}\|_a < \gamma$ exists, if inequalities

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m_w} < \gamma^2 \qquad (10)$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & B_w^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ B_w & -\Pi & 0 \\ D_{zw} & 0 & -I_{p_z} \end{bmatrix} \prec 0 \qquad (11)$$

$$\begin{bmatrix} -\Pi & 0 & \Pi A^{\mathrm{T}} + \Lambda^{\mathrm{T}} B_u^{\mathrm{T}} & \Pi C_z^{\mathrm{T}} + \Lambda^{\mathrm{T}} D_{zu}^{\mathrm{T}} \\ 0 & -\eta I_{m_w} & B_w^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ A\Pi + B_u\Lambda & B_w & -\Pi & 0 \\ C_z\Pi + D_{zu}\Lambda & D_{zw} & 0 & -I_{p_z} \end{bmatrix} \prec 0 \qquad (12)$$

$$\eta > \gamma^2 \qquad \Psi \succ 0 \qquad \Pi \succ 0 \qquad (13)$$

are solvable w.r.t. η, real $(m_w \times m_w)$-matrix Ψ, $(n_x \times n_x)$-matrix Π and $(m_u \times n_x)$-matrix Λ. If problem (10)–(13) is solvable and unknown variables are found, then $K = \Lambda\Pi^{-1}$

Corollary: Inequalities (10)–(13) are convex in Ψ, affine in Π and Λ, and also linear in γ^2. Minimizing γ^2 under convex constraints (10)–(13), we minimize γ. Denote $\widehat{\gamma} := \gamma^2$. Conditions of Theorem 3 allows minimize γ by solving convex optimization problem

$$\begin{gathered} \widehat{\gamma} \longrightarrow \inf \\ \text{over} \quad \eta, \Psi, \Pi, \Lambda, \widehat{\gamma} \quad \text{that satisfy (10)–(13)} \end{gathered} \tag{14}$$

If problem (14) is solvable, state-feedback anisotropic controller is computed similar to Theorem 3: $K = \Lambda\Pi^{-1}$

Anisotropic γ-optimal controller

Anisotropic controllers computed by solving convex optimization problems similar to (14) are called anisotropic *γ-optimal* controllers

Standard LDTI state-space model:

$$P(z):\ \begin{bmatrix} x_{k+1} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix}$$

$x_k \in \mathbb{R}^{n_x}$, $w_k \in \mathbb{R}^{m_w}$, $u_k \in \mathbb{R}^{m_u}$, $z_k \in \mathbb{R}^{p_z}$, $y_k \in \mathbb{R}^{p_y}$, $\overline{\mathbf{A}}(W) \leqslant a$
Dynamic output-feedback controller with state $\xi_k \in \mathbb{R}^{n_\xi}$:

$$K(z):\ \begin{bmatrix} \xi_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} A_{\mathrm{c}} & B_{\mathrm{c}} \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{bmatrix} \begin{bmatrix} \xi_k \\ y_k \end{bmatrix} \qquad (15)$$

Kimura's condition of order n_ξ :

$$n_\xi > n_x - m_u - p_y$$

Problem 2. Synthesis of fixed-order output-feedback controller

Given $a \geqslant 0$ $\gamma > 0$, find output-feedback controller (15) of fixed-order n_ξ that stabilizes closed-loop system ($\rho(\mathcal{A}) < 1$) and ensures $\|T_{zw}\|_a < \gamma$

Apply conditions (6), (7) of Theorem 2 to closed-loop realization

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A + B_u D_c C_y & B_u C_c & B_w + B_u D_c D_{yw} \\ B_c C_y & A_c & B_c D_{yw} \\ \hline C_z + D_{zu} D_c C_y & D_{zu} C_c & D_{zw} + D_{zu} D_c D_{yw} \end{array}\right] \qquad (16)$$

Corollary. Solution to fixed-order controller synthesis problem

Given $a \geqslant 0$ and $\gamma > 0$, controller $K(z)$ of fixed-order n_ξ that stabilizes closed-loop system ($\rho(\mathcal{A}) < 1$) and ensures $|||T_{zw}|||_a < \gamma$ exists, if inequalities (18)–(21) are solvable w.r.t. η, real $(m_w \times m_w)$-matrix Ψ, controller realization matrices $A_c \in \mathbb{R}^{n_\xi \times n_\xi}$, $B_c \in \mathbb{R}^{n_\xi \times p_y}$, $C_c \in \mathbb{R}^{m_u \times n_\xi}$, $D_c \in \mathbb{R}^{m_u \times p_y}$ and two reciprocal $(n \times n)$-matrices Φ, Π,

$$\Phi\Pi = I_n \qquad (17)$$

where $n = n_x + n_\xi$

Corollary. Solution to fixed-order controller synthesis problem

$$\eta - (\mathrm{e}^{-2a}\det\Psi)^{1/m_w} < \gamma^2 \tag{18}$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * & * \\ B_w + B_u D_c D_{yw} & -\Pi_{11} & * & * \\ B_c D_{yw} & -\Pi_{12}^{\mathrm{T}} & -\Pi_{22} & * \\ D_{zw} + D_{zu} D_c D_{yw} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{19}$$

$$\begin{bmatrix} -\Phi_{11} & * & * & * & * & * \\ -\Phi_{12}^{\mathrm{T}} & -\Phi_{22} & * & * & * & * \\ 0 & 0 & -\eta I_{m_w} & * & * & * \\ A + B_u D_c C_y & B_u C_c & B_w + B_u D_c D_{yw} & -\Pi_{11} & * & * \\ B_c C_y & A_c & B_c D_{yw} & -\Pi_{12}^{\mathrm{T}} & -\Pi_{22} & * \\ C_z + D_{zu} D_c C_y & D_{zu} C_c & D_{zw} + D_{zu} D_c D_{yw} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{20}$$

$$\eta > \gamma^2 \qquad \Psi \succ 0 \qquad \Phi := \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^{\mathrm{T}} & \Phi_{22} \end{bmatrix} \succ 0 \qquad \Pi := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^{\mathrm{T}} & \Pi_{22} \end{bmatrix} \succ 0 \tag{21}$$

$$\Phi\Pi = I_n$$

From block partitioning in (21) and condition (17) it follows that

$$\Phi \begin{bmatrix} \Pi_{11} \\ \Pi_{12}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} I_{n_x} \\ 0 \end{bmatrix} \qquad \Phi\Pi_1 = \Phi_1 \qquad \Pi\Phi_1 = \Pi_1$$

$$\Phi_1 := \begin{bmatrix} I_{n_x} & \Phi_{11} \\ 0 & \Phi_{12}^{\mathrm{T}} \end{bmatrix} \qquad \Pi_1 := \begin{bmatrix} \Pi_{11} & I_{n_x} \\ \Pi_{12}^{\mathrm{T}} & 0 \end{bmatrix} \tag{22}$$

It can be verified by direct calculation:

$$\Pi_1^{\mathrm{T}}\Phi\Pi_1 = \Phi_1^{\mathrm{T}}\Pi_1 = \Phi_1^{\mathrm{T}}\Pi\Phi_1 = \Pi_1^{\mathrm{T}}\Phi_1 = \begin{bmatrix} \Pi_{11} & I_{n_x} \\ I_{n_x} & \Phi_{11} \end{bmatrix} \tag{23}$$

Linearizing change of variables [Gahinet, 1996]

$$\mathcal{A}_{\mathrm{c}} := \Phi_{12}A_{\mathrm{c}}\Pi_{12}^{\mathrm{T}} + \Phi_{12}B_{\mathrm{c}}C_y\Pi_{11} + \Phi_{11}B_uC_{\mathrm{c}}\Pi_{12}^{\mathrm{T}} + \Phi_{11}(A + B_uD_{\mathrm{c}}C_y)\Pi_{11} \tag{24}$$

$$\mathcal{B}_{\mathrm{c}} := \Phi_{12}B_{\mathrm{c}} + \Phi_{11}B_uD_{\mathrm{c}} \tag{25}$$

$$\mathcal{C}_{\mathrm{c}} := C_{\mathrm{c}}\Pi_{12}^{\mathrm{T}} + D_{\mathrm{c}}C_y\Pi_{11} \tag{26}$$

$$\mathcal{D}_{\mathrm{c}} := D_{\mathrm{c}} \tag{27}$$

Theorem 4. Solution to full-order output-feedback controller synthesis problem

Given $a \geqslant 0$ and $\gamma > 0$, dynamic output-feedback controller K of full order $n_\xi = n_x$ such that $\rho(\mathcal{A}) < 1$ and $\|T_{zw}\|_a < \gamma$ exists, if inequalities (28)–(31) are solvable w.r.t. η, real $(m_w \times m_w)$-matrix Ψ, matrices $\mathcal{A}_\mathrm{c} \in \mathbb{R}^{n_x \times n_x}$, $\mathcal{B}_\mathrm{c} \in \mathbb{R}^{n_x \times p_y}$, $\mathcal{C}_\mathrm{c} \in \mathbb{R}^{m_u \times n_x}$, $\mathcal{D}_\mathrm{c} \in \mathbb{R}^{m_u \times p_y}$ and two $(n_x \times n_x)$-matrices Π_{11}, Φ_{11}. If problem (28)–(31) is solvable and unknown variables are found, then controller realization matrices $A_\mathrm{c} \in \mathbb{R}^{n_x \times n_x}$, $B_\mathrm{c} \in \mathbb{R}^{n_x \times p_y}$, $C_\mathrm{c} \in \mathbb{R}^{m_u \times n_x}$, $D_\mathrm{c} \in \mathbb{R}^{m_u \times p_y}$ are uniquely defined by

$$
\begin{aligned}
D_\mathrm{c} &:= \mathcal{D}_\mathrm{c} \\
C_\mathrm{c} &:= (\mathcal{C}_\mathrm{c} - D_\mathrm{c} C_y \Pi_{11}) \Pi_{12}^{-\mathrm{T}} \\
B_\mathrm{c} &:= \Phi_{12}^{-1} (\mathcal{B}_\mathrm{c} - \Phi_{11} B_u D_\mathrm{c}) \\
A_\mathrm{c} &:= \Phi_{12}^{-1} (\mathcal{A}_\mathrm{c} - \Phi_{12} B_\mathrm{c} C_y \Pi_{11} - \Phi_{11} B_u C_\mathrm{c} \Pi_{12}^{\mathrm{T}} - \Phi_{11}(A + B_u D_\mathrm{c} C_y) \Pi_{11}) \Pi_{12}^{-\mathrm{T}}
\end{aligned}
$$

where two nonsingular $(n_x \times n_x)$-matrices Π_{12}, Φ_{12} satisfy equation

$$\Pi_{12} \Phi_{12}^{\mathrm{T}} = I_{n_x} - \Pi_{11} \Phi_{11}$$

Theorem 4. Solution to full-order output-feedback controller synthesis problem

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m_w} < \gamma^2 \tag{28}$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * & * \\ B_w + B_u \mathcal{D}_c D_{yw} & -\Pi_{11} & * & * \\ \Phi_{11} B_w + \mathcal{B}_c D_{yw} & -I_{n_x} & -\Phi_{11} & * \\ D_{zw} + D_{zu} \mathcal{D}_c D_{yw} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{29}$$

$$\begin{bmatrix} -\Pi_{11} & * & * & * & * & * \\ -I_{n_x} & -\Phi_{11} & * & * & * & * \\ 0 & 0 & -\eta I_{m_w} & * & * & * \\ A\Pi_{11} + B_u \mathcal{C}_c & A + B_u \mathcal{D}_c C_y & B_w + B_u \mathcal{D}_c D_{yw} & -\Pi_{11} & * & * \\ \mathcal{A}_c & \Phi_{11} A + \mathcal{B}_c C_y & \Phi_{11} B_w + \mathcal{B}_c D_{yw} & -I_{n_x} & -\Phi_{11} & * \\ C_z \Pi_{11} + D_{zu} \mathcal{C}_c & C_z + D_{zu} \mathcal{D}_c C_y & D_{zw} + D_{zu} \mathcal{D}_c D_{yw} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{30}$$

$$\eta > \gamma^2 \quad \Pi_{11} \succ 0 \quad \Phi_{11} \succ 0 \quad \begin{bmatrix} \Pi_{11} & I_{n_x} \\ I_{n_x} & \Phi_{11} \end{bmatrix} \succ 0 \tag{31}$$

Corollary: Inequalities (28)–(31) are linear in $\widehat{\gamma} := \gamma^2$. Conditions of Theorem 4 allows minimum of threshold value γ to be found from solving convex optimization problem

$$\begin{gathered} \widehat{\gamma} \to \inf \\ \text{over} \quad \eta, \Psi, \Phi_{11}, \Pi_{11}, \mathcal{A}_{\mathrm{c}}, \mathcal{B}_{\mathrm{c}}, \mathcal{C}_{\mathrm{c}}, \mathcal{D}_{\mathrm{c}}, \widehat{\gamma} \quad \text{that satisfy (28)–(31)} \end{gathered} \tag{32}$$

If problem (32) is solvable, anisotropic γ-optimal full-order controller is computed similar to Theorem 4

Standard LDTI state-space model:

$$P(z): \begin{bmatrix} x_{k+1} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix} \tag{33}$$

$x_k \in \mathbb{R}^{n_x}$, $w_k \in \mathbb{R}^{m_w}$, $u_k \in \mathbb{R}^{m_u}$, $z_k \in \mathbb{R}^{p_z}$, $y_k \in \mathbb{R}^{p_y}$, $\overline{\mathbf{A}}(W) \leqslant a$
Static output-feedback controller:

$$u_k = K y_k \tag{34}$$

Zero-order Kimura's condition: $n_x - m_u - p_y < 0$

Problem 3. Synthesis of static output-feedback controller

Given $a \geqslant 0$ and $\gamma > 0$, find SOF controller (34) that stabilizes closed-loop system

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{c|c} A + B_u K C_y & B_w + B_u K D_{yw} \\ \hline C_z + D_{zu} K C_y & D_{zw} + D_{zu} K D_{yw} \end{array}\right]$$

i.e. $\rho(\mathcal{A}) < 1$, and ensures $|||T_{zw}|||_a < \gamma$ holds true

Corollary. Solution to general SOF controller synthesis problem

Given $a \geqslant 0$, $\gamma > 0$, controller (34) exists, if inequalities

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m_w} < \gamma^2$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * \\ B_w + B_u K D_{yw} & -\Pi & * \\ D_{zw} + D_{zu} K D_{yw} & 0 & -I_{p_z} \end{bmatrix} \prec 0$$

$$\begin{bmatrix} -\Phi & * & * & * \\ 0 & -\eta I_{m_w} & * & * \\ A + B_u K C_y & B_w + B_u K D_{yw} & -\Pi & * \\ C_z + D_{zu} K C_y & D_{zw} + D_{zu} K D_{yw} & 0 & -I_{p_z} \end{bmatrix} \prec 0$$

$$\eta > \gamma^2 \quad \Psi \succ 0 \quad \Phi \succ 0 \quad \Pi \succ 0$$

are solvable w.r.t. η, real $(m_w \times m_w)$-matrix Ψ, $(m_u \times p_y)$-matrix K and two reciprocal $(n_x \times n_x)$-matrices Φ, Π,

$$\Phi \Pi = I_{n_x}$$

Linearizing change of variables [Scherer, 2000] works for a class of plants with structural property

$$P_{yu}(z) := C_y(zI - A)^{-1}B_u = 0 \tag{35}$$

If plant with property (35) is stabilizable and detectable ⇒ there exists a nonsingular coordinate transformation matrix T s.t.

$$\left[\begin{array}{c|cc} TAT^{-1} & TB_w & TB_u \\ \hline C_zT^{-1} & D_{zw} & D_{zu} \\ C_yT^{-1} & D_{yw} & 0 \end{array}\right] = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{w_1} & B_{u_1} \\ 0 & A_{22} & B_{w_2} & 0 \\ \hline C_{z_1} & C_{z_2} & D_{zw} & D_{zu} \\ 0 & C_{y_2} & D_{yw} & 0 \end{array}\right]$$

(A_{11}, B_{u_1}) controllable, (A_{11}, C_{y_2}) observable, A_{22} stable

Closed-loop system realization:

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} + B_{u_1}KC_{y_2} & B_{w_1} + B_{u_1}KD_{yw} \\ 0 & A_{22} & B_{w_2} \\ \hline C_{z_1} & C_{z_2} + D_{zu}KC_{y_2} & D_{zw} + D_{zu}KD_{yw} \end{array}\right] \tag{36}$$

Lyapunov matrix Φ is decomposed to blocks according to representation of $\mathcal{A}$ in (36):

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^{\mathrm{T}} & \Phi_{22} \end{bmatrix} \succ 0$$

Linearizing change of variables [Scherer, 2000]

$$P := \begin{bmatrix} Q & S \\ S^{\mathrm{T}} & R \end{bmatrix} = \begin{bmatrix} \Phi_{11}^{-1} & -\Phi_{11}^{-1}\Phi_{12} \\ -\Phi_{12}^{\mathrm{T}}\Phi_{11}^{-1} & \Phi_{22} - \Phi_{12}^{\mathrm{T}}\Phi_{11}^{-1}\Phi_{12} \end{bmatrix} \tag{37}$$

Backward transformation [Scherer, 2000]

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^{\mathrm{T}} & \Phi_{22} \end{bmatrix} = \begin{bmatrix} Q^{-1} & -Q^{-1}S \\ -S^{\mathrm{T}}Q^{-1} & R - S^{\mathrm{T}}Q^{-1}S \end{bmatrix}$$

Transformation (37) is motivated by factorization

$$P_1\Phi = P_2$$

$$P_1 := \begin{bmatrix} Q & 0 \\ S^{\mathrm{T}} & I \end{bmatrix} \qquad P_2 := \begin{bmatrix} I & -S \\ 0 & R \end{bmatrix}$$

Theorem 5. Solution to synthesis problem when $P_{yu}(z) = 0$

Let plant (33) satisfy (35). Given $a \geqslant 0$, $\gamma > 0$, SOF anisotropic controller (34) exists if inequalities

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m_w} < \gamma^2 \tag{38}$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & * & * & * \\ B_{w_1} + B_{u_1} K D_{yw} - S B_{w_2} & -Q & * & * \\ R B_{w_2} & 0 & -R & * \\ D_{zw} + D_{zu} K D_{yw} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{39}$$

$$\begin{bmatrix} -Q & * & * & * & * & * \\ 0 & -R & * & * & * & * \\ 0 & 0 & -\eta I_{m_w} & * & * & * \\ A_{11} Q & A_{11} S - S A_{22} + A_{12} + B_{u_1} K C_{y_2} & B_{w_1} + B_{u_1} K D_{yw} - S B_{w_2} & -Q & * & * \\ 0 & R A_{22} & R B_{w_2} & 0 & -R & * \\ C_{z_1} Q & C_{z_1} S + C_{z_2} + D_{zu} K C_{y_2} & D_{zw} + D_{zu} K D_{yw} & 0 & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{40}$$

$$\eta > \gamma^2 \quad \Psi \succ 0 \quad Q \succ 0 \quad R \succ 0 \tag{41}$$

are solvable w.r.t. η, real $(m_w \times m_w)$-matrix Ψ, controller matrix K and matrices Q, R, S

Other two particular plant structure cases leads the SOF controller synthesis problem to convex optimization problem: singular control and filtering problems [e.g. Lee, Lee, Kwon, 2006]

Singular control problem: $D_{zu} = 0$, $\operatorname{rank} B_u = m_u$, closed-loop realization:

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{c|c} A + B_u K C_y & B_w + B_u K D_{yw} \\ \hline C_z & D_{zw} \end{array}\right]$$

$\Rightarrow$ There exists a nonsingular coordinate transformation matrix T_u s.t.

$$\bar{B}_u := T_u B_u = \begin{bmatrix} I_{m_u} \\ 0 \end{bmatrix}$$

In this basis

$$\bar{A} := T_u A T_u^{-1} \qquad \bar{B}_w := T_u B_w \qquad \bar{C}_z := C_z T_u^{-1} \qquad \bar{C}_y := C_y T_u^{-1}$$

Theorem 6. Solution to singular control problem

Let plant (33) satisfy $D_{zu} = 0$, $\operatorname{rank} B_u = m_u$. Given $a \geqslant 0$, $\gamma > 0$, SOF anisotropic controller (34) exists if inequalities

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m_w} < \gamma^2 \tag{42}$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & \bar{B}_w^{\mathrm{T}} \bar{S}^{\mathrm{T}} + D_{yw}^{\mathrm{T}} L^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ \bar{S}\bar{B}_w + L D_{yw} & \bar{\Phi} - \bar{S} - \bar{S}^{\mathrm{T}} & 0 \\ D_{zw} & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{43}$$

$$\begin{bmatrix} -\bar{\Phi} & 0 & \bar{A}^{\mathrm{T}} \bar{S}^{\mathrm{T}} + \bar{C}_y^{\mathrm{T}} L^{\mathrm{T}} & \bar{C}_z^{\mathrm{T}} \\ 0 & -\eta I_{m_w} & \bar{B}_w^{\mathrm{T}} \bar{S}^{\mathrm{T}} + D_{yw}^{\mathrm{T}} L^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ \bar{S}\bar{A} + L\bar{C}_y & \bar{S}\bar{B}_w + L D_{yw} & \bar{\Phi} - \bar{S} - \bar{S}^{\mathrm{T}} & 0 \\ \bar{C}_z & D_{zw} & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{44}$$

$$\eta > \gamma^2 \qquad \Psi \succ 0 \qquad \bar{\Phi} \succ 0 \tag{45}$$

are solvable w.r.t. η, $(m_w \times m_w)$-matrix Ψ, $(n_x \times n_x)$-matrix $\bar{\Phi}$ and structured matrix variables $\bar{S} := \begin{bmatrix} \bar{S}_1 & 0 \\ 0 & \bar{S}_2 \end{bmatrix}$, $L := \begin{bmatrix} L_1 \\ 0 \end{bmatrix}$, at that $K = \bar{S}_1^{-1} L_1$

Singular filtering problem: $D_{yw} = 0$, rank $C_y = p_y$, closed loop realization:

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{c|c} A + B_u K C_y & B_w \\ \hline C_z + D_{zu} K C_y & D_{zw} \end{array}\right]$$

There exists a nonsingular coordinate transformation matrix T_y s.t.

$$\bar{C}_y := C_y T_y^{-1} = \left[\begin{array}{cc} I_{p_y} & 0 \end{array}\right]$$

In this basis

$$\bar{A} := T_y A T_y^{-1} \qquad \bar{B}_w := T_y B_w \qquad \bar{B}_u := T_y B_u \qquad \bar{C}_z := C_z T_y^{-1}$$

Theorem 7. Solution to singular filtering problem

Let plant (33) satisfy $D_{yw} = 0$, $\operatorname{rank} C_y = p_y$. Given $a \geqslant 0$, $\gamma > 0$, SOF anisotropic controller (34) exists if inequalities

$$\eta - (\mathrm{e}^{-2a} \det \Psi)^{1/m_w} < \gamma^2 \tag{46}$$

$$\begin{bmatrix} \Psi - \eta I_{m_w} & \bar{B}_w^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ \bar{B}_w & -\bar{\Pi} & 0 \\ D_{zw} & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{47}$$

$$\begin{bmatrix} \bar{\Pi} - \bar{R} - \bar{R}^{\mathrm{T}} & 0 & \bar{R}^{\mathrm{T}}\bar{A}^{\mathrm{T}} + M^{\mathrm{T}}\bar{B}_u^{\mathrm{T}} & \bar{R}^{\mathrm{T}}\bar{C}_z^{\mathrm{T}} + M^{\mathrm{T}}D_{zu}^{\mathrm{T}} \\ 0 & -\eta I_{m_w} & \bar{B}_w^{\mathrm{T}} & D_{zw}^{\mathrm{T}} \\ \bar{A}\bar{R} + \bar{B}_u M & \bar{B}_w & -\bar{\Pi} & 0 \\ \bar{C}_z\bar{R} + D_{zu}M & D_{zw} & 0 & -I_{p_z} \end{bmatrix} \prec 0 \tag{48}$$

$$\eta > \gamma^2 \quad \Psi \succ 0 \quad \bar{\Pi} \succ 0 \tag{49}$$

are solvable w.r.t. η, $(m_w \times m_w)$-matrix Ψ, $(n_x \times n_x)$-matrix $\bar{\Pi}$ and structured matrix variables $\bar{R} := \begin{bmatrix} \bar{R}_1 & 0 \\ 0 & \bar{R}_2 \end{bmatrix}$, $M := \begin{bmatrix} M_1 & 0 \end{bmatrix}$, at that $K = M_1 \bar{R}_1^{-1}$

Fixed-order controller synthesis problem can be embedded into SOF controller synthesis problem for plant with extended state:

$$\begin{bmatrix} \mathcal{A} & \mathcal{B}_w & \mathcal{B}_u \\ \mathcal{C}_z & \mathcal{D}_{zw} & \mathcal{D}_{zu} \\ \mathcal{C}_y & \mathcal{D}_{yw} & 0 \end{bmatrix} := \left[\begin{array}{cc|c|cc} A & 0 & B_w & 0 & B_u \\ 0 & 0 & 0 & I_{n_\xi} & 0 \\ \hline C_z & 0 & D_{zw} & 0 & D_{zu} \\ \hline 0 & I_{n_\xi} & 0 & 0 & 0 \\ C_y & 0 & D_{yw} & 0 & 0 \end{array}\right]$$

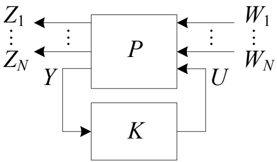
Closed-loop realization

$$\begin{aligned} \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathfrak{C} & \mathfrak{D} \end{bmatrix} &= \begin{bmatrix} \mathcal{A} & \mathcal{B}_w \\ \mathcal{C}_z & \mathcal{D}_{zw} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_u \\ \mathcal{D}_{zu} \end{bmatrix} K \begin{bmatrix} \mathcal{C}_y & \mathcal{D}_{yw} \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{A} + \mathcal{B}_u K \mathcal{C}_y & \mathcal{B}_w + \mathcal{B}_u K \mathcal{D}_{zw} \\ \mathcal{C}_z + \mathcal{D}_{zw} K \mathcal{C}_y & \mathcal{D}_{zw} + \mathcal{D}_{zu} K \mathcal{D}_{yw} \end{bmatrix} \end{aligned}$$

K joins the controller parameters:

$$K := \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$$

- Standard linear plant model P
- Linear controller K
- Groups of external input channels W_j (commands, disturbances, noises), $\overline{\mathbf{A}}(W_j) \leqslant a_j$
- Control U
- Groups of controlled outputs Z_j
- Measurement Y



Problem 4. Multichannel synthesis problem

Given $a_j \geqslant 0$, $\gamma_j > 0$ find controller $K(z)$ of order n_ξ that stabilizes closed-loop system and ensures

$$|||T_{z_j w_j}|||_{a_i} < \gamma_j \qquad j = 1 \ldots N \tag{50}$$

hold true simultaneously, where $T_{z_j w_j} = \mathcal{L}_j T_{zw} \mathcal{R}_j$, $\mathcal{L}_j$, $\mathcal{R}_j$ are input/output selection matrices

Convex optimization problems:

- state-feedback controller
- full-order output-feedback controller
- static output-feedback controller for plants that satisfy $P_{yu}(z) = 0$

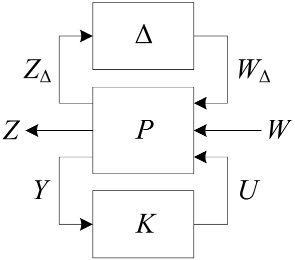
Allow to place closed-loop poles in arbitrary LMI region. Anisotropic norm can be applied as performance criterion in general setting of multiobjective control [Scherer, Gahinet, Chilali, 1996], [Scherer, 2000]

Problems of searching for reciprocal matrices that satisfy convex constraints:

- fixed-order output-feedback controller
- static output-feedback controller (general case)

Allow to place closed-loop poles in circle of given radius

- Standard linear plant model P
- Linear controller K
- External input W: $\overline{\mathbf{A}}(W) \leqslant a$
- Control U
- Controlled output Z
- Measurement Y
- Uncertainty $\Delta_k \in \triangle$: $\Delta_k^{\mathrm{T}} \Delta_k \leqslant \gamma_\Delta^2 I$
- $W_\Delta = \Delta_k Z_\Delta$



Problem 5. Synthesis of controller for system with uncertainty

Given $a \geqslant 0$, $\gamma_\Delta, \gamma > 0$ find controller $K(z)$ of order n_ξ that stabilizes closed-loop system and ensures

$$\left\| T_{zw} \right\|_a < \gamma \qquad (51)$$

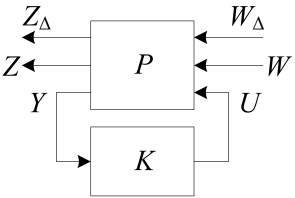
holds true for all admissible uncertainties $\Delta_k \in \triangle$: $\Delta_k^{\mathrm{T}} \Delta_k \leqslant \gamma_\Delta^2 I$

Consider auxiliary system
where uncertainty input W_Δ and output Z_Δ are related by

$$\mathbf{E}|w_{\Delta k}|^2 \leqslant \gamma_\Delta^2 \mathbf{E}|z_{\Delta k}|^2 \quad \forall k$$

with extended controlled output

$$\widetilde{Z} = \begin{bmatrix} Z_\Delta \\ Z \end{bmatrix}$$



Problem 6. Auxiliary synthesis problem

Given $a \geqslant 0$, $\gamma_\Delta, \gamma > 0$ find controller $K(z)$ of order n_ξ that stabilizes closed-loop system and ensures

$$\|T_{\tilde{z}w}\|_a < \gamma \qquad (52)$$

$$\|T_{\tilde{z}w_\Delta}\|_\infty < \gamma_\Delta^{-1} \qquad (53)$$

hold simultaneously

Theorem 8. Connection between auxiliary and main problems

Let controller K solve auxiliary Problem 6, i.e. inequalities (52), (53) hold true for closed-loop system $\mathcal{F}_l(\widetilde{P}, K)$. Then inquality (51) holds true for closed-loop system $\mathcal{F}_l(P_\Delta, K)$ with the same controller K for all $\Delta \in \triangle$, i.e. controller K also solves main Problem 5. Converse is generally incorrect.

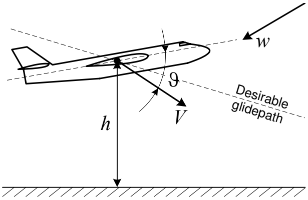
Convex optimizatation problems:

- state-feedback controller,
- full-order output-feedback controller

Allow to place closed-loop poles in arbitrary LMI region robustly

Problems of searching for reciprocal matrices that satisfy convex constraints:

- fixed-order output-feedback controller
- static output-feedback controller (general case)



- Controlled variables:
  - altitude h
  - airspeed V
- Control:
  - engine thrust
  - elevators
- Disturbance: W

$$
\begin{aligned}
m\dot{V} &= T\cos\alpha - D - mg\sin\theta - m(\dot{w}_x\cos\theta + \dot{w}_y\sin\theta)\\
mV\dot{\theta} &= T\sin\alpha + L - mg\cos\theta + m(\dot{w}_x\sin\theta - \dot{w}_y\cos\theta)\\
J_z\dot{\omega}_z &= M_z\\
\dot{\vartheta} &= \omega_z\\
\dot{h} &= V\sin\theta + w_y\\
\dot{T} &= \frac{1}{T_e}(-T + K_e\delta_t)
\end{aligned}
$$

$$\delta_e = K_{\omega_z}\omega_z + K_\vartheta\vartheta + K_{cy}\vartheta_{cy}, \quad \alpha = \vartheta - \theta$$

Control objective

Minimize influence of disturbance on deviation of altitude and airspeed

Discrete-time linearized model (standard plant):

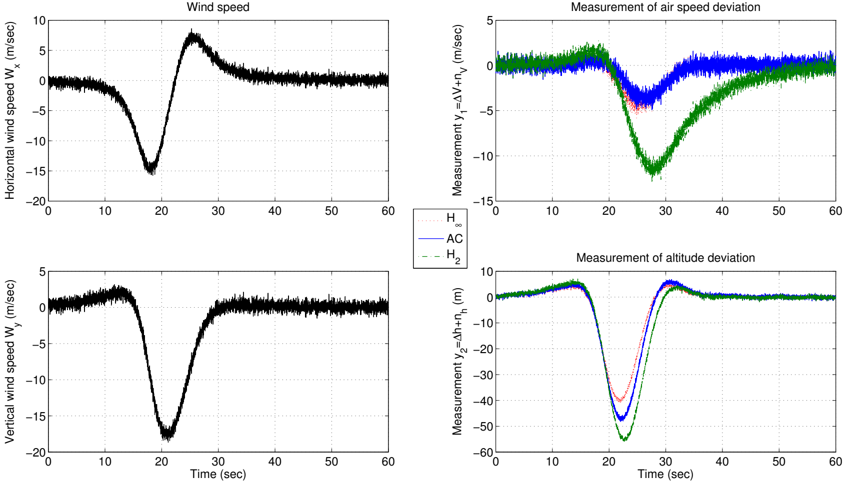
$$P(z):\ \begin{bmatrix} x_{k+1} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_k \\ u_k \end{bmatrix}$$

$$\begin{aligned} x_k &= \begin{bmatrix} \Delta V_k, & \Delta\theta_k, & \Delta\omega_{z,k}, & \Delta\vartheta_k, & \Delta h_k, & \Delta T_k \end{bmatrix}^{\mathrm{T}} \\ w_k &= \begin{bmatrix} w_{y,k}, & \dot{w}_{x,k}, & \dot{w}_{y,k}, & n_{y_1,k}, & n_{y_2,k} \end{bmatrix}^{\mathrm{T}},\ \overline{\mathbf{A}}(W) \leqslant a \\ u_k &= \begin{bmatrix} \Delta\theta_{cy,k}, & \Delta\delta_{t,k} \end{bmatrix}^{\mathrm{T}} \\ z_k &= \begin{bmatrix} \Delta V_k, & \Delta h_k, & \Delta\vartheta_{cy,k}, & \Delta\delta_{t,k} \end{bmatrix}^{\mathrm{T}} \\ y_k &= \begin{bmatrix} \Delta V_k + n_{y_1,k}, & \Delta h_k + n_{y_2,k} \end{bmatrix}^{\mathrm{T}} \end{aligned}$$

Problem statement

Given $a \geqslant 0$ and $\gamma > 0$, find (generally dynamic) output-feedback controller (15) to stabilize closed-loop system ($\rho(\mathcal{A}) < 1$) and ensure $|\!|\!| T_{zw} |\!|\!|_a < \gamma$ holds true

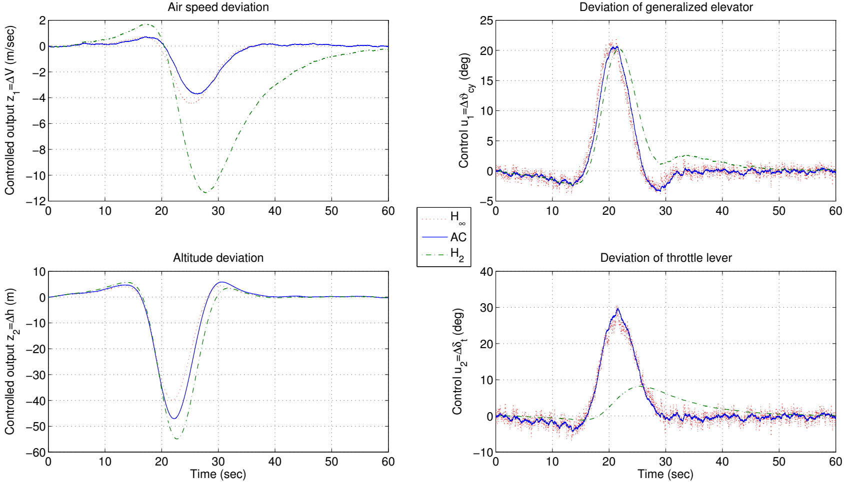
Full-order output-feedback controllers



Horizontal and vertical components of wind speed W_x, W_y

Measurements of airspeed $\Delta V + n_V$ and altitude $\Delta h + n_h$

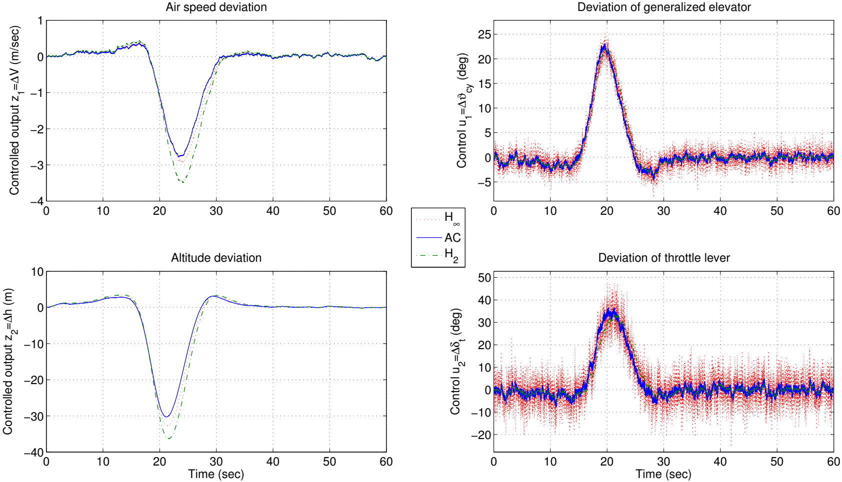
Full-order output-feedback controllers



Controlled output: airspeed ΔV, altitude Δh

Control: elevator $\Delta\vartheta_{cy}$, throttle lever $\Delta\delta_t$

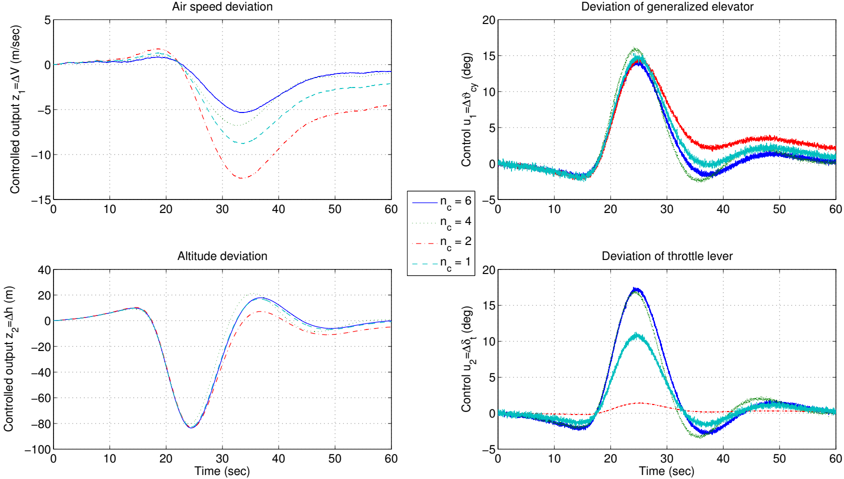
Full-order output-feedback controllers with pole placement



Controlled output: airspeed ΔV, altitude Δh

Control: elevator $\Delta\vartheta_{cy}$, throttle lever $\Delta\delta_t$

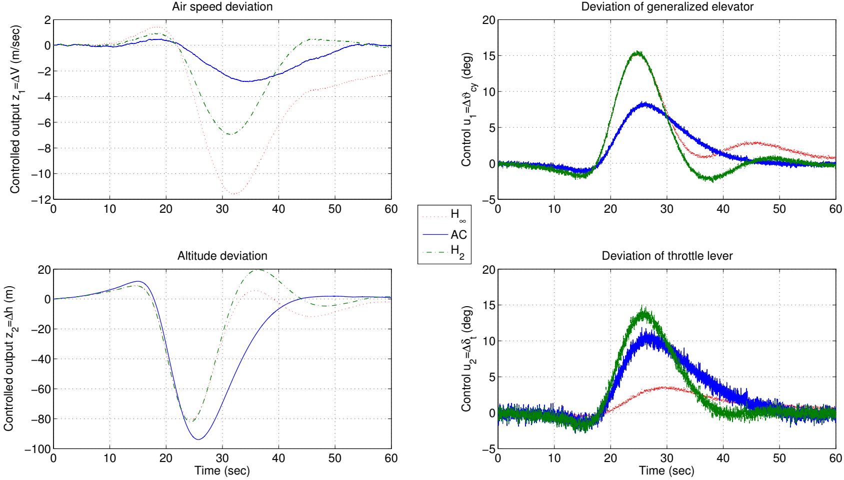
Fixed-order output-feedback controllers with pole placement in circle



Controlled output: airspeed ΔV, altitude Δh

Control: elevator $\Delta\vartheta_{cy}$, throttle lever $\Delta\delta_t$

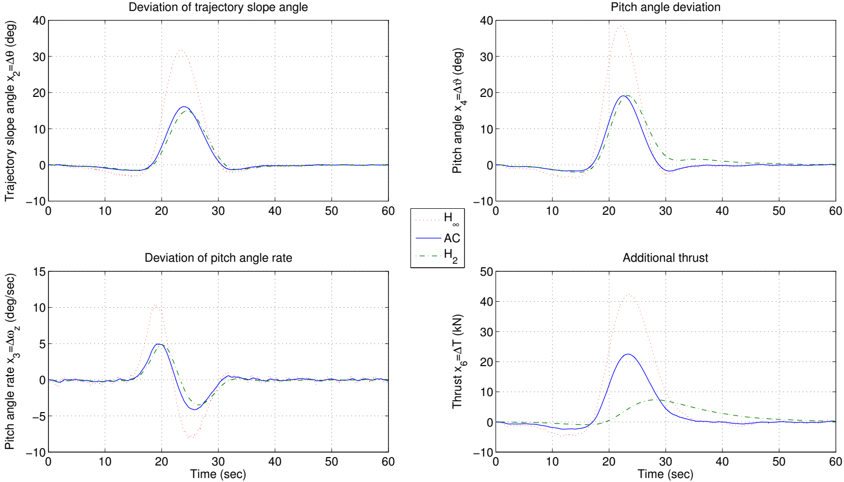
Static output feedback controllers with pole placement in circle



Controlled output: airspeed ΔV, altitude Δh

Control: elevator $\Delta\vartheta_{cy}$, throttle lever $\Delta\delta_t$

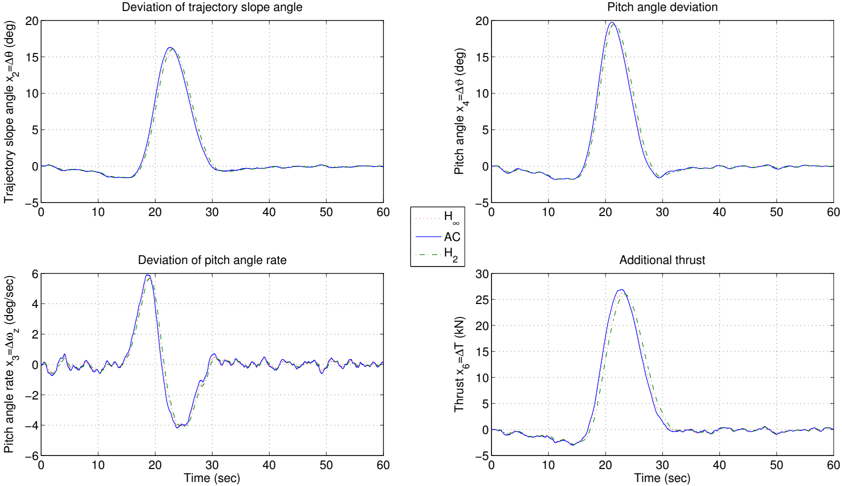
Full-order output-feedback controllers



Trajectory slope angle $\Delta\theta$, pitch rate $\Delta\omega_z$

Pitch $\Delta\vartheta$, engine thrust ΔT

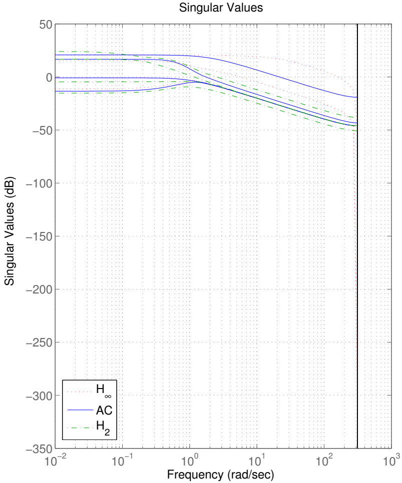
Full-order output-feedback controllers with pole placement



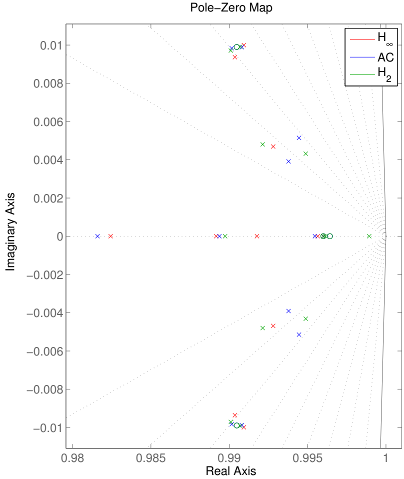
Trajectory slope angle $\Delta\theta$, pitch rate $\Delta\omega_z$

Pitch $\Delta\vartheta$, engine thrust ΔT

Full-order output-feedback controllers

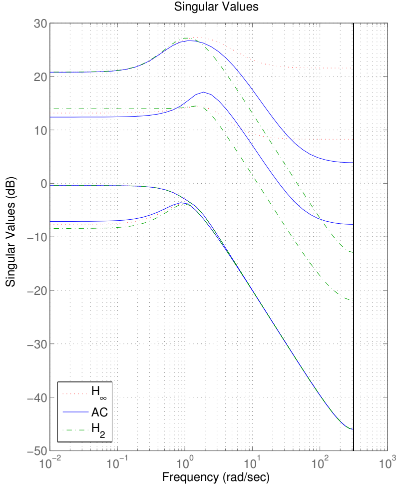


Singular values

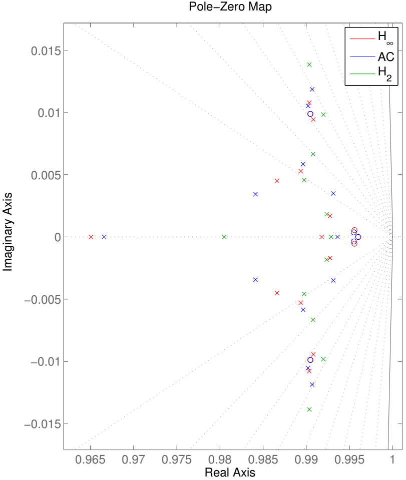


Closed-loop pole-zero map

Full-order output-feedback controllers with pole placement



Singular values



Closed-loop pole-zero map

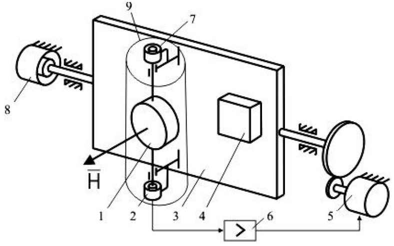
Closed loop with full-order controller

	Controller K_2	Controller K_a	Controller K_∞
Solution results:			
$\gamma^\star$	0.516	5.4203	10.894
$\|\|T_{zw}\|\|_2$	0.516	1.1473	3.1448
$\|\|\|T_{zw}\|\|\|_{0.7}$	7.8391	5.1768	5.5944
$\|\|T_{zw}\|\|_\infty$	15.855	10.93	10.891
Simulation results:			
max $\|\Delta V\|$, m/s	11.3	3.559	4.329
max $\|\Delta h\|$, m	54.79	46.87	39.79
max $\|\Delta \theta\|$, grad	14.86	16.04	31.6
max $\|\Delta \omega_z\|$, grad/s	4.884	5.043	10.56
max $\|\Delta \vartheta\|$, grad	19.06	19	38.08
max $\|\Delta T\|$, kN	7.263	22.58	42.48
max $\|\Delta \vartheta_{cy}\|$, grad	20.7	20.8	21.91
max $\|\Delta \delta_t\|$, grad	8.224	29.25	29.23

Closed loop with pole placing full-order controller

	Controller K_2	Controller K_a	Controller K_∞
Solution results:			
$\gamma^\star$	2.4714	1.6461	23.17
$\|\|T_{zw}\|\|_2$	2.4058	3.4063	12.598
$\|\|\|T_{zw}\|\|\|_{0.7}$	1.1489	1.5681	5.6791
$\|\|T_{zw}\|\|_\infty$	22.9	21.655	23.112
Simulation results:			
max $\|\Delta V\|$, m/s	3.427	2.716	2.872
max $\|\Delta h\|$, m	36.4	30.34	32.8
max $\|\Delta \theta\|$, grad	16.02	16.25	16.07
max $\|\Delta \omega_z\|$, grad/s	5.825	6.205	5.863
max $\|\Delta \vartheta\|$, grad	19.54	19.72	19.39
max $\|\Delta T\|$, kN	26.17	27.42	26.53
max $\|\Delta \vartheta_{cy}\|$, grad	22.01	23.22	25.44
max $\|\Delta \delta_t\|$, grad	35.25	38.65	54.01

Monoaxial powered gyrostabilizer

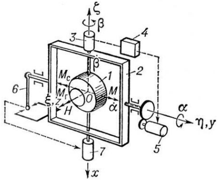


1. gyroscope
2. sensor of angle β of SE GU precession axis
3. gyrostabilized platform
4. object of stabilization
5. force stabilization motor of GSP
6. stabilization loop amplifier
7. sensor of moments
8. sensor of angle α of GSP stabilization axis
9. gyro unit

Control objective

Track input commands α_c defining angular position α of gyrostabilized platform stabilization axis and simultaneously stabilize to zero angular position β of sensor element of gyro unit under unknown external disturbing torques and weakly correlated noises of measurements of angular velocities

Model of monoaxial gyrostabilizer considers effect of oscillations of rotor of synchronous hysteresis-reluctance motor (SHM) to angle errors



$$
\begin{aligned}
\dot{\alpha}(t) &= \omega_\alpha(t) \\
\dot{\omega}_\alpha(t) &= -\frac{K_{g\alpha}}{J_\alpha}\omega_\alpha(t) - \frac{H(t)}{J_\alpha}\omega_\beta(t) + \frac{1}{J_\alpha}M_\alpha^w(t) - \frac{1}{J_\alpha}M_\alpha^u(t) \\
\dot{\beta}(t) &= \omega_\beta(t) \\
\dot{\omega}_\beta(t) &= \frac{H(t)}{J_\beta}\omega_\alpha(t) - \frac{K_{g\beta}}{J_\beta}\omega_\beta(t) + \frac{1}{J_\beta}M_\beta^w(t) - \frac{1}{J_\beta}M_\beta^u(t)
\end{aligned}
$$

- $\alpha(t)$, $\beta(t)$: angular positions of GSP stabilization axis and SE GU precession axis
- $\omega_\alpha(t)$, $\omega_\beta(t)$: angular velocities of GSP and SE GU
- $H(t) = H_0 + \Delta H_0 \sin(2\pi f t)$: variable kinetic moment of GU
  - H_0 : nominal kinetic moment of GU
  - ΔH_0 : amplitude of harmonic change of KM GU
  - f : oscillation frequency of rotor of SHM
- J_α, J_β : moments of inertia of GSP and SE GU
- $K_{g\alpha}$, $K_{g\beta}$: coefficients of viscous friction
- $M_\alpha^w(t)$, $M_\beta^w(t)$: generalized external disturbing torques
- $M_\alpha^u(t)$, $M_\beta^u(t)$: control actions of GSP and GU force stabilization motors

To solve angular position control problem, equation system is expanded and put into standard form

$$\widetilde{P}(z): \begin{bmatrix} x_{k+1} \\ z_{\Delta k} \\ z_k \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_\Delta & B_w & B_u \\ C_\Delta & 0 & D_{\Delta w} & D_{\Delta u} \\ C_z & 0 & D_{zw} & D_{zu} \\ C_y & 0 & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_{\Delta k} \\ w_k \\ u_k \end{bmatrix} \qquad z_{\Delta k} = \Delta w_{\Delta k}$$

$$x_k = \begin{bmatrix} \int(\alpha_c - \alpha) & \alpha & \omega_\alpha & \int\beta & \beta & \omega_\beta & M_\alpha^u & M_\beta^u \end{bmatrix}^{\mathrm{T}}$$

$$w_{1k} = \begin{bmatrix} \alpha_c & M_\alpha^w & M_\beta^w \end{bmatrix}^{\mathrm{T}} \qquad w_{2k} = \begin{bmatrix} n_\alpha & n_\beta \end{bmatrix}^{\mathrm{T}} \qquad u_k = \begin{bmatrix} u_\alpha & u_\beta \end{bmatrix}^{\mathrm{T}}$$

$$z_k = \begin{bmatrix} \int(\alpha_c - \alpha) & \int\beta & u_\alpha & u_\beta \end{bmatrix}^{\mathrm{T}}$$

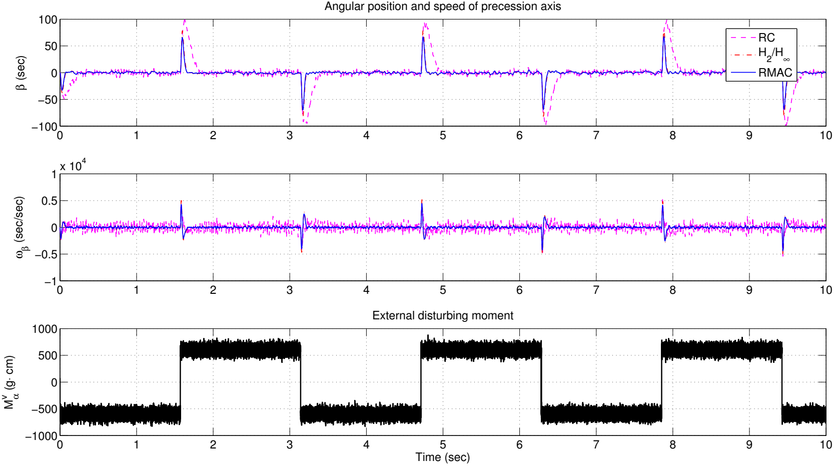
$$y_k = \begin{bmatrix} \int(\alpha_c - \alpha) & \alpha & \omega_\alpha + n_\alpha & \int\beta & \beta & \omega_\beta + n_\beta \end{bmatrix}^{\mathrm{T}}$$

Multiobjective robust controller synthesis problem

Given $a_1 \geqslant 0$, $a_2 \geqslant 0$, $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_\Delta > 0$, find dynamic output-ffedback controller such that $\rho(\mathcal{A}) < 1$ and

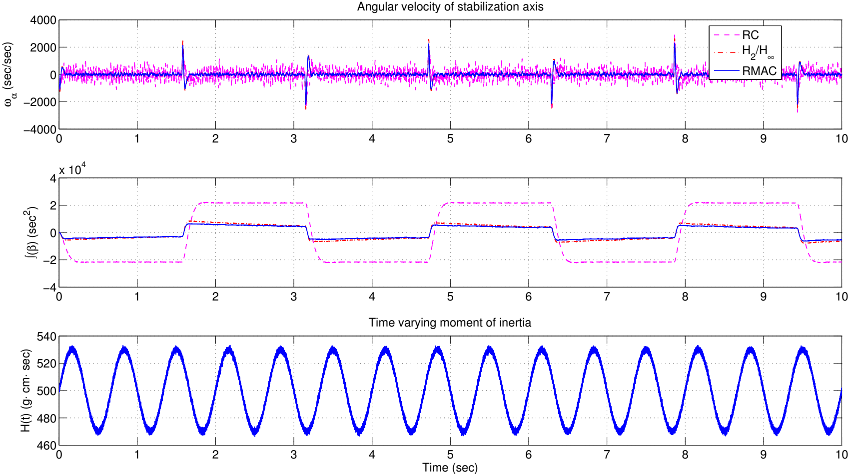
$$|\!|\!| T_{zw_1} |\!|\!|_{a_1} < \gamma_1 \qquad |\!|\!| T_{zw_2} |\!|\!|_{a_2} < \gamma_2 \qquad \forall \Delta\colon \overline{\sigma}(\Delta) \leqslant \gamma_\Delta^{-1}$$

Stabilization of GSP under disturbances



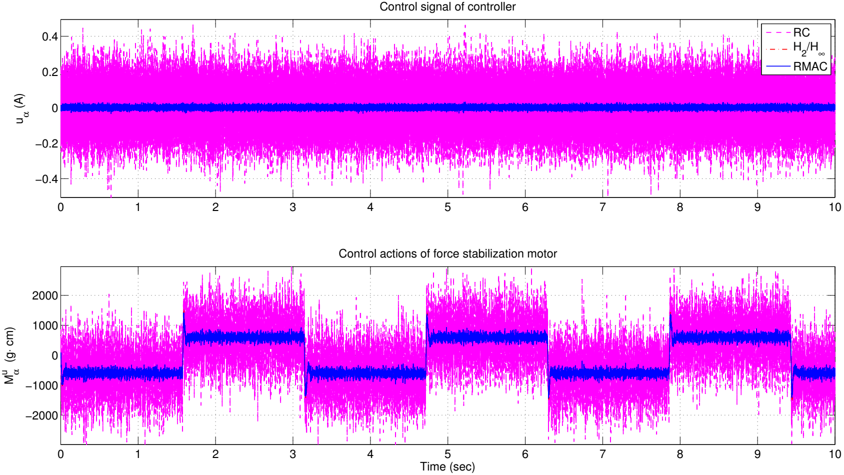
Angular position β and velocity ω_β of SE GU precession axis; external disturbing torque M_α^w

Stabilization of GSP under disturbances



Angular velocity ω_α of GSP stabilization axis; integral of angular position of SE GU precession axis $\int \beta$; variable kinetic moment H of GU

Stabilization of GSP under disturbances

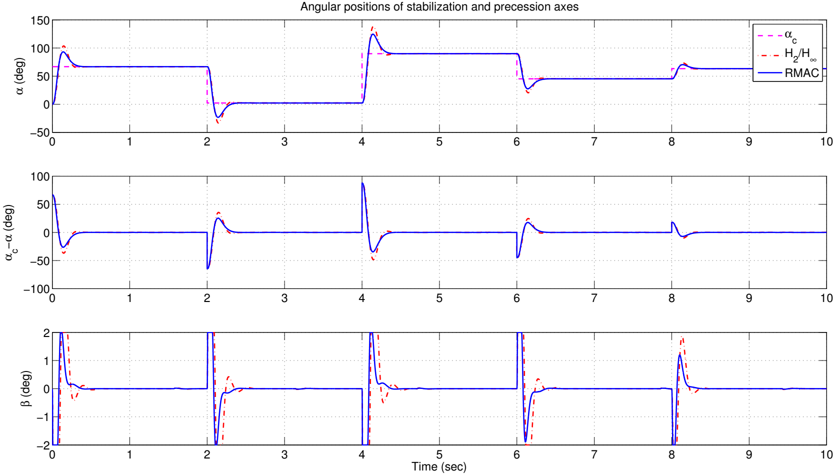


Control signal u_α of controller;
control action M^u_α of GSP force stabilization motor

Stabilization of GSP under disturbances: closed loop with different controllers

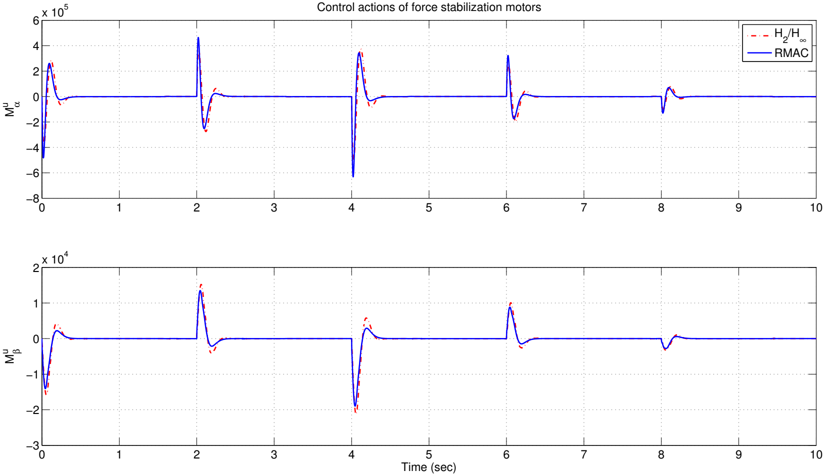
	Controller		
	K	K_{a_1/a_2}	$K_{\infty/2}$
Solution results:			
$\gamma^\star_{a_1}$ $(\gamma^\star_\infty)$	—	0.12727	0.057519
$\gamma^\star_{a_2}$ $(\gamma^\star_2)$	—	0.19681	0.29955
γ_Δ	—	60	60
$\lvert\lvert\lvert T_{zw_1} \rvert\rvert\rvert_{1.45}$	0.92164	0.061381	0.083662
$\lvert\lvert\lvert T_{zw_2} \rvert\rvert\rvert_{0.05}$	0.1768	0.020269	0.019157
$\lVert T_{zw_1} \rVert_\infty$	$3.4078 \cdot 10^{-5}$	$4.0384 \cdot 10^{-5}$	$4.6281 \cdot 10^{-5}$
$\lVert T_{zw_2} \rVert_2$	0.11792	0.0080756	0.0040558
Simulation results:			
max $\lvert\beta\rvert$, ”	99.33	69.65	81.99
max $\lvert\omega_\alpha\rvert$, ”/s	2889	2272	2627
max $\lvert\omega_\beta\rvert$, ”/s	5370	4463	5172
max $\lvert M^u_\alpha\rvert$, g·cm	3942	1431	1545
max $\lvert u_\alpha\rvert$, A	0.54	0.03496	0.01655

Tracking GSP angular position



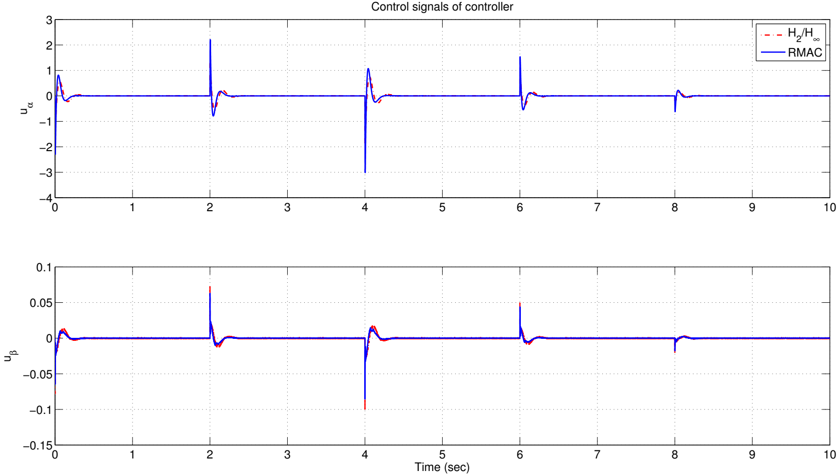
Angular position α of GSP stabilization axis; tracking error e_α; angular position β of SE GU precession axis

Tracking GSP angular position



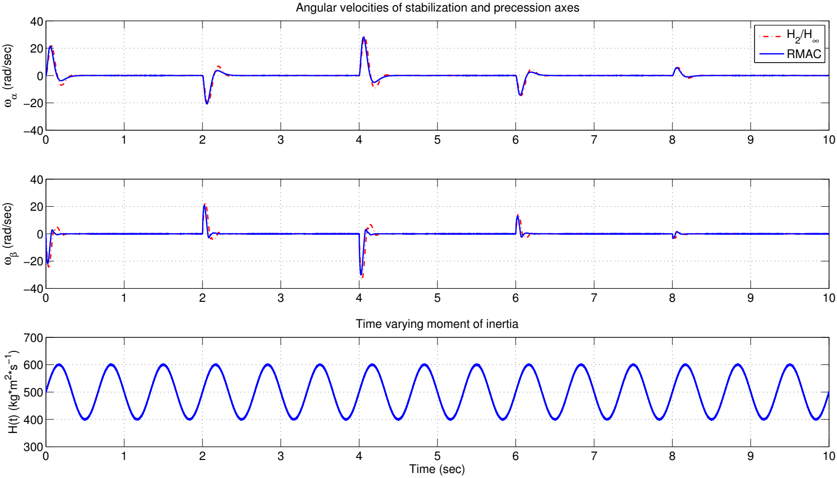
Control actions M^u_α and M^u_β of force stabilization motors of GSP and SE GU

Tracking GSP angular position



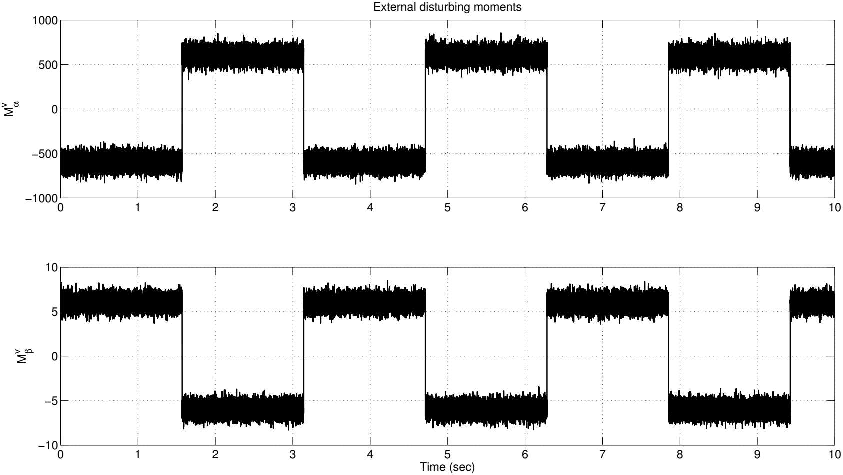
Control signals u_α and u_β of controller

Tracking GSP angular position



Angular velocities ω_α, ω_β; variable kinetic moment H of GU

Tracking GSP angular position



Disturbing torques M_α^v, M_β^v

Tested models from *COMPl$_e$ib* collection [Leibfritz, 2004]:

- aircrafts: transport, supersonic, VTOL
- helicopters: transport, military
- jet engines
- stabilization of optical system
- cruise missile (“terrain-following model”), “air-to-air” missile
- Euler-Bernoulli beam (order 10. . . 80)
- distillation binary tower
- electric power system
- jet pack
- submersible vehicle

Results:

1. Bounded Real Lemma for anisotropic norm in terms of inequalities
2. Synthesis of fixed-order anisotropic suboptimal controllers in form of dynamic output-feedback compensators via semidefinite programming and numerical optimization
3. Synthesis of static output-feedback anisotropic suboptimal controllers
4. Synthesis of anisotropic γ-optimal controllers via convex optimization
5. Solution to multiobjective problems of anisotropic control
6. Synthesis of anisotropic suboptimal controllers that ensure closed-loop pole placement in given LMI region of complex plain
7. Synthesis of robust anisotropic controllers for models with uncertain parameters

- Tchaikovsky M.M. and Kurdyukov A.P. On computing anisotropic norm of linear discrete-time-invariant system via LMI-based approach // *Archives of Control Sciences*, 2006, Vol. 16, No. 3, p. 257–281.
- Tchaikovsky M.M., Kurdyukov A.P., and Timin V.N. Strict anisotropic norm bounded real lemma in terms of inequalities // *Proc. 18th IFAC World Congr.*, Milano, Italy, August 28–September 2, 2011, p. 2332–2337.
- Tchaikovsky M.M., Kurdyukov A.P. Strict anisotropic norm bounded real lemma in terms of matrix inequalities // *Doklady Mathematics*, 2011, Vol. 84, No. 3, p. 895–898.
- Tchaikovsky M.M. Static output feedback anisotropic controller design by LMI-based approach: General and special cases // *Proc. 2012 American Control Conf.*, Montreal, Canada, June 27–29, 2012, p. 5208–5213.
- Timin V.N., Tchaikovsky M.M., and Kurdyukov A.P. A solution to anisotropic suboptimal filtering problem by convex optimization // *Doklady Mathematics*, 2012, Vol. 85, No. 3, p. 443–445.
- Tchaikovsky M.M., Kurdyukov A.P., and Nikiforov V.M. LMI-Based design of multichannel anisotropic suboptimal controllers with application to control of gyrostabilized platform // To appear at *IEEE Multi-Conference on Systems and Control*, Dubrovnik, Croatia, October 3–5, 2012.

Moc vám děkuji za pozornost!