Analysis of the solution map governed by a parametrized differential inclusion

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Outline

Model

- General system

$$F(t, x(t), \dot{x}(t)) \in \Lambda(t), \;\; t \in [0, T] \; a.e.$$
$$x(0) = a$$

- $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ a continuously differentiable function
- $\Lambda : [0, T] \rightrightarrows \mathbb{R}^m$ a multifunction independent of the state variable x
- If $\Lambda(t) = \{0\}$ and $F = \dot{x}(t) - f(t, x(t))$, we obtain an ODE

$$\dot{x}(t) = f(t, x(t)).$$

Typical properties

- Sweeping process

$$-\dot{x}(t) + f(t, x(t)) \in \mathrm{N}_{\Gamma(t)}(x(t))$$
$$x(0) = a$$

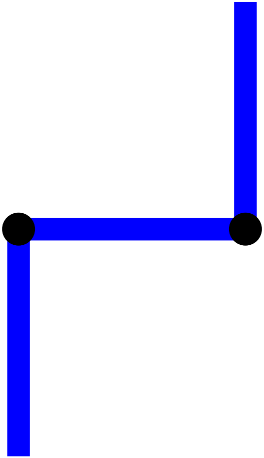
- Desired reformulation

$$\underbrace{\begin{pmatrix} x(t) \\ -\dot{x}(t) + f(t, x(t)) \end{pmatrix}}_{\displaystyle\downarrow \; F(t, x(t), \dot{x}(t))} \quad \in \quad \underbrace{\operatorname{gph} \mathrm{N}_{\Gamma(t)}}_{\displaystyle\downarrow \; \Lambda(t)}$$

- ∇F has artificial rows. The full row rank property of ∇F is usually not available.

- Λ is not regular.

Two kinks for $\Lambda = \mathrm{gph}_{\mathrm{N}_{[0,1]}}$



Introduction of control

- Controlled system 1

$$F(t, u(t), x(t), \dot{x}(t)) \in \Lambda(t), \;\; t \in [0, T] \; a.e.$$
$$x(0) = a$$

- Controlled system 2

$$F(t, u, x(t), \dot{x}(t)) \in \Lambda(t), \;\; t \in [0, T] \; a.e.$$
$$x(0) = a$$

- u control variable
- x state variable
- Goal: analysis of the solution map $S : u \mapsto x$, known also as control–to–state operator.

Application 1

- Electrical circuit

$$-A_1(u)\dot{x}(t) - A_0(u)x(t) + f(t) \in \mathrm{N}_{\Gamma(t)}(\dot{x}(t))$$

- A_1, A_0 parameters of various components of the circuit
- $x(t)$ current on these components

Application 2

- Lower level of an dynamic MPEC (Mathematical program with equilibrium constraints)

$$\begin{aligned} &\min\ J(u,x) \\ &s.t.\ x \in \operatorname*{argmin}_{x' \in K} L(u,x') \\ &\quad u \in \Omega \end{aligned}$$

- Karush–Kuhn–Tucker form

$$\begin{aligned} &\min\ J(u,x) \\ &s.t.\ 0 \in \nabla_x L(u,x) + \mathrm{N}_K(x) \\ &\quad u \in \Omega \end{aligned}$$

- Natural occurrence of gph N.

Normal cone

- Painlewé–Kuratowski upper limit

$$\operatorname{Limsup}_n A_n = \{x;\ \exists x_n \in A_n; x \text{ is an accumulation point of } \{x_n\}\}$$

- Normal cone

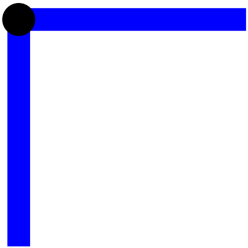
$$\hat{\mathrm{N}}_A(x) = \{x^*; \langle x^*, x' - x\rangle \le o(\|x' - x\|) \text{ for all } x' \in A\}$$
$$\mathrm{N}_A(x) = \operatorname{Limsup}_{x' \xrightarrow{A} x} \hat{\mathrm{N}}_A(x')$$
$$\bar{\mathrm{N}}_A(x) = \operatorname{cl}\operatorname{co}\mathrm{N}_A(x).$$

- A convex

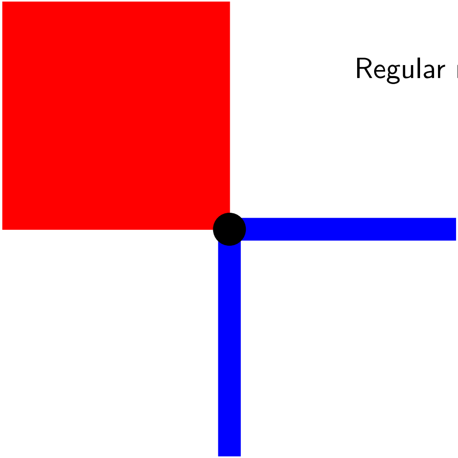
$$\mathrm{N}_A(x) = \{x^*; \langle x^*, x' - x\rangle \le 0 \text{ for all } x' \in A\}.$$

- If A has $\mathcal{C}^1$ boundary, all cones are exactly one ray.

Differences between cones

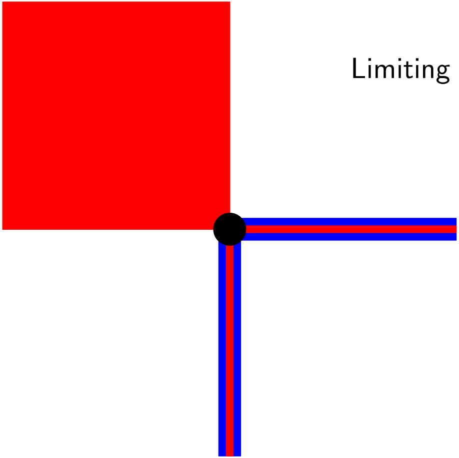


Differences between cones



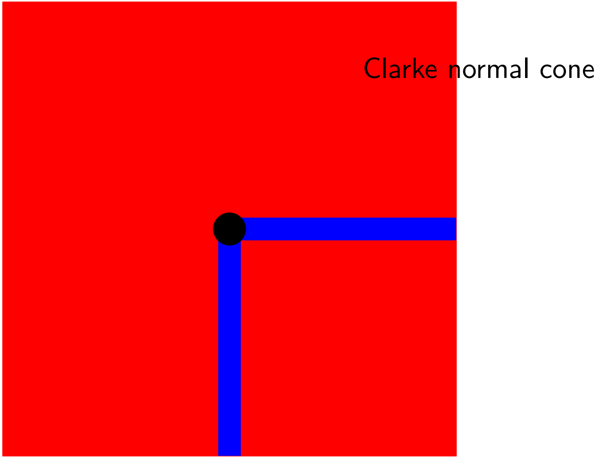
Regular normal cone

Differences between cones



Limiting normal cone

Differences between cones



Subdifferential

- Definition

$$\hat{\partial} f(x) = \{x^*; (x^*, -1) \in \hat{\mathrm{N}}_{\mathrm{epi}\, f}(x, f(x))\}$$
$$\partial f(x) = \{x^*; (x^*, -1) \in \mathrm{N}_{\mathrm{epi}\, f}(x, f(x))\}$$
$$\bar{\partial} f(x) = \{x^*; (x^*, -1) \in \bar{\mathrm{N}}_{\mathrm{epi}\, f}(x, f(x))\}$$

- If f convex, then all subdifferentials are equal to the subdifferential in convex sense.
- If f is differentiable, then $\hat{\partial} f(x) = \{\nabla f(x)\}$ but $\partial f(x) \supset \{\nabla f(x)\}$.
- If f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$.

Coderivative

- Subdifferential uses the ordering on $\mathbb{R}$. Unfortunately, this is not possible if f is multivalued or maps to $\mathbb{R}^m$.
- For $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we define coderivative $D^*M : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ as

$$D^*M(x,y)(y^*) = \{x^*; (x^*, -y^*) \in \mathrm{N}_{\mathrm{gph}\, M}(x,y)\}.$$

- If M is single–valued and continuously differentiable, then

$$D^*M(x)(y^*) = D^*M(x, M(x))(y^*) = (\nabla M(x))^T y^*$$

Aubin property

- $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ satisfies the Aubin property at (u, x) if there are neighborhoods U of u and V of x and a positive number L such that

$$M(\tilde{u}) \cap V \subset M(\hat{u}) + L\|\tilde{u} - \hat{u}\|\mathbb{B}$$

  for all $\tilde{u}, \hat{u} \in U$.
- Localization not only in domain but also in range.
- For M single–valued the Aubin property coincides with the local Lipschitzian property.

Discretization

- Controlled system 1

$$f_k(u_k, x_k, x_{k+1}) \in \Lambda_k, \; k = 0, \ldots, K-1 \\ x_0 = a \tag{1}$$

- $S : \mathbb{R}^{Kd} \to \mathbb{R}^{Kn}$
- Controlled system 2

$$f_k(u, x_k, x_{k+1}) \in \Lambda_k, \; k = 0, \ldots, K-1 \\ x_0 = a \tag{2}$$

- $S : \mathbb{R}^{d} \to \mathbb{R}^{Kn}$

Comparison of both systems

- The estimate of coderivative has a very similar form for both systems.
- For time–dependent control u_k it may be much simpler to verify the used constraint qualification.
- This implies that for an optimal control problem for time–independent control u it may be advantegous to add artificial variables u_k and set
$$u_1 = \cdots = u_K.$$
- For time–independent control u it is possible to pass to a limit and obtain local Lipschitzian property even in the continuous case.

Applications of coderivative

Theorem

Consider $S : \mathbb{R}^n \to \mathbb{R}^m$ and consider any $x \in S(u)$. Suppose that $\operatorname{gph} S$ is locally closed at (u, x). Then S has the Aubin property around (u, x) if and only if $D^*S(u, x)(0) = \{0\}$ and in this case

$$lipS(u,x) = \|D^*S(u,x)\|^* := \sup_{\|x^*\|=1} \sup_{u^* \in D^*S(u,x)(x^*)} \|u^*\|.$$

Theorem

Let $f(x) = g(F(x))$ with $F : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ both Lipschitz continuous at x. Then

$$\partial f(x) \subset \bigcup_{y^* \in \partial g(F(x))} D^*F(x)(y^*).$$

Necessary optimality conditions

- Optimal control problem

$$
\begin{aligned}
&\min\ J(u,x)\\
&s.t.\ f_k(u_k,x_k,x_{k+1})\in\Lambda_k\\
&\quad\ u\in\Omega.
\end{aligned}
$$

- Assume that S is single–valued. Then

$$
\begin{aligned}
&\min\ J(u,S(u))\\
&\quad\ u\in\Omega.
\end{aligned}
$$

- Necessary optimality conditions

$$
\begin{aligned}
0&\in\partial(J\circ S)(u)+\mathrm{N}_\Omega(u)\\
&\subset\partial_u J(u,S(u))+D^*S(u)(\partial_x J(u,S(u)))+\mathrm{N}_\Omega(u)
\end{aligned}
$$

Constraint qualification 1

- System reformulation

$$\begin{array}{ccc} f_0(u_0, x_0, x_1) & \in & \Lambda_0 \\ \dots & & \dots \\ f_{K-1}(u_{K-1}, x_{K-1}, x_K) & \in & \Lambda_{K-1} \\ \downarrow & & \downarrow \\ F(u, x) & & \Omega \end{array}$$

- Implicit function theorem: $\nabla_x F$ has full row rank
- Used CQ is implied by: ∇F has full row rank
- Weaker CQ but also weaker results (only Lipschitzian continuity of S).

Constraint qualification 2

If there exist multipliers

$$p_k \in \mathrm{N}_{\Lambda_k}(f_k(u_k, x_k, x_{k+1})),\ \ k = 0, \ldots, K-1 \tag{3}$$

satisfying the following conditions

$$\text{Problem (1)} : 0 = (\nabla_u f_k)^T p_k,\ \ k = 0, \ldots, K-1$$

$$\text{Problem (2)} : 0 = \sum_{k=0}^{K-1} (\nabla_u f_k)^T p_k$$

and

$$0 = (\nabla_v f_{k-1})^T p_{k-1} + (\nabla_x f_k)^T p_k,\ \ k = 1, \ldots, K$$

$$0 = (\nabla_v f_{K-1})^T p_{K-1}.$$

Then $p_k = 0$.

Theorem

Consider problem (1) with f_k continuously differentiable and Λ_k closed. Then for any

$$u^* \in D^* S(u,x)(x^*) \in \mathbb{R}^{Kd}$$

with $u^* = (u_0^*, \ldots, u_{K-1}^*)$ and $x^* = (x_1^*, \ldots, x_K^*)$ there exist multipliers (3) such that

$$u_k^* = (\nabla_u f_k)^T p_k.$$

Moreover, the following terminal condition and the adjoint equations are satisfied.

$$\begin{aligned} -x_K^* &= (\nabla_v f_{K-1})^T p_{K-1} \\ -x_k^* &= (\nabla_v f_{k-1})^T p_{k-1} + (\nabla_x f_k)^T p_k, \ k = 1, \ldots, K \end{aligned} \tag{4}$$

Corollary

Consider problem (2) and let the assumptions of the previous theorem be fulfilled. Then for any

$$u^* \in D^* S(u,x)(x^*) \in \mathbb{R}^d$$

there are multipliers (3) such that

$$u^* = \sum_{k=0}^{K-1} (\nabla_u f_k)^T p_k.$$

Moreover, the terminal condition and adjoint equations (4) are satisfied.

Sensitivity analysis

Theorem

If in the setting of the previous corollary it holds that

$$\|{u^*}^K\|_2 \leq L(K)\|{x^*}^K\|_2,$$

then S^K has the Aubin property with modulus $L(K)$.
Further, assume that S^K and S are single–valued. Fix any $u \in \mathbb{R}^d$ and $\varepsilon > 0$, set $V := B(u, \varepsilon)$ and define

$$M(K, \varepsilon) := \sup_{\tilde{u} \in V} L(K, \tilde{u})$$

to be the Lipschitzian modulus of S^K on V.

Theorem (continued)

Further consider piecewise constant or piecewise linear extension of $S^K(\tilde{u})$ and assume that $S^K(\tilde{u}) \rightharpoonup S(\tilde{u})$ in $L^2([0,T],\mathbb{R}^n)$ for all $\tilde{u} \in V$. If

$$M(\varepsilon) := \operatorname{limsup} \frac{1}{\sqrt{K}} M(K,\varepsilon) < \infty,$$

then S is locally Lipschitz at V with modulus $\sqrt{T}M(\varepsilon)$.

Example

- Consider the first application problem

$$-A_1(u)\dot{y}(t) - A_0(u)y(t) + f(t) \in \mathrm{N}_{C(t)}(\dot{y}(t))$$

- Perform a discretization

$$\begin{aligned} -A_1(u)z_{k+1}^K - A_0(u)(y_k^K + h^K z_{k+1}^K) + f_{k+1}^K &\in \mathrm{N}_{C_{k+1}^K}(z_{k+1}^K) \\ y_{k+1}^K - y_k^K - h^K z_{k+1}^K &= 0. \end{aligned}$$

- Omit upper indeces and rewrite it into a desired form

$$\begin{aligned} \begin{pmatrix} z_{k+1} \\ -A_1(u)z_{k+1} - A_0(u)(y_k + hz_{k+1}) + f_{k+1} \end{pmatrix} &\in \operatorname{gph} \mathrm{N}_{C_{k+1}} \\ y_{k+1} - y_k - hz_{k+1} &= 0. \end{aligned}$$

- Set u to be the control variable and $x = (y, z)$ the state variable.

Example (continued)

- Coderivative estimate

$$u^* = -\sum_{k=1}^{K} [\nabla_u A_0(u)(y_k + hz_{k+1}) + \nabla_u A_1(u) z_{k+1}]^T q_k.$$

- Adjoint equations and terminal condition

$$\begin{aligned} p_k - (hA_0 + A_1) q_k &= hr_k - y_k^* \\ r_k &= r_{k+1} + A_0 q_{k+1} - z_k^* \\ p_K - (hA_0 + A_1) q_K &= hr_K - y_K^* \\ r_K &= -z_K^*. \end{aligned}$$

- And multipliers

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} \in \mathrm{N}_{\mathrm{gph}\, \mathrm{N}_{C_k}}(\cdot)$$

$$r_k \text{ free.}$$

Lemma

Let A be a positive definite matrix. Consider the following equation

$$p - Aq = r$$

which is to be solved for known r with respect to p and q satisfying $p^T q \leq 0$.
Denoting

$$d := \min_{|x|=1} x^T Ax,$$

then for any p and q solving the equation and satisfying the constraint, one has

$$|q| \leq \frac{1}{d}|r|.$$

Lemma

Assume that $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *is a maximal monotone mapping. Then for every* $(x, y) \in \operatorname{gph} T$ *and every*

$$\begin{pmatrix} p \\ q \end{pmatrix} \in \mathrm{N}_{\operatorname{gph} T}(x, y)$$

one has $p^T q \leq 0$*.*

Applied for

$$T(x) = \mathrm{N}_C(x),$$

which is a maximal monotone mapping for convex C.

Example (continued)

- Assumptions
  - A_i positive definite
  - Continuous differentiability of $u \mapsto A_i(u)$
  - $C(t)$ convex for all t
- After some computation

  $$\|u^{*K}\|_2 \le bc\max\{cT|A_0|e^{c|A_0|T}+1,\, Te^{c|A_0|T}\}\sqrt{2Kn}\|(y^{*K}, z^{*K})\|_2$$

  for some constants b, c.
- Hence S^K is Lipschitz continuous with modulus

  $$bc\max\{cT|A_0|e^{c|A_0|T}+1,\, Te^{c|A_0|T}\}\sqrt{2Kn}.$$

Example (continued)

- Under additional assumptions we obtain

$$y^K \rightrightarrows y$$
$$z^K \rightharpoonup \dot{y} \text{ in } L^2$$

- Hence assumptions of the main theorem are fulfilled and

$$S : u \mapsto (y, \dot{y})$$
$$\mathbb{R}^d \to L^2([0, T], \mathbb{R}^n) \times L^2([0, T], \mathbb{R}^n)$$

  is Lipschitz continuous

- Equivalently

$$S : u \mapsto y$$
$$\mathbb{R}^d \to W^{12}([0, T], \mathbb{R}^n)$$

  is Lipschitz continuous.

Final notes

- Applied in this field rather unused method for sensitivity analysis of a parametrized differential inclusion.
- This method is particularly suitable for sweeping process.
- Managed to derive conditions for Lipschitz continuity of

$$S : \mathbb{R}^d \to W^{12}([0, T], \mathbb{R}^n).$$

- Computed the Lipschitzian modulus.

Future plans

- Create a more general framework and incorporate more possible problem classes.
- Continuous control variable.
- Infinite–dimensional range space.